

Theory of Computer Science

D4. Halting Problem Variants & Rice's Theorem

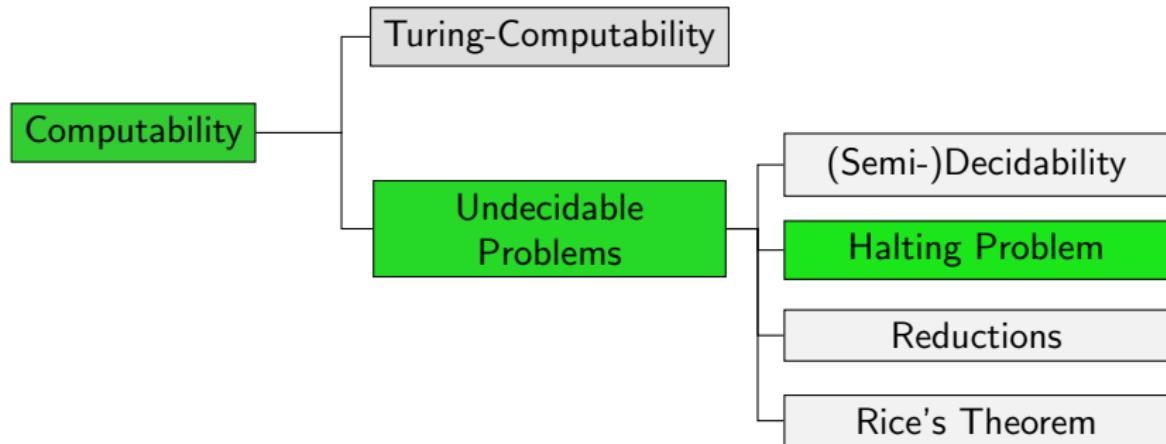
Gabriele Röger

University of Basel

April 29, 2020

Other Halting Problem Variants

Overview: Computability Theory



Reminder: Special Halting Problem

Definition (Special Halting Problem)

The **special halting problem** or **self-application problem** is the language

$$K = \{w \in \{0, 1\}^* \mid M_w \text{ started on } w \text{ terminates}\}.$$

German: spezielles Halteproblem, Selbstanwendbarkeitsproblem

General Halting Problem (1)

Definition (General Halting Problem)

The **general halting problem** or **halting problem** is the language

$$H = \{w\#x \in \{0, 1, \#\}^* \mid w, x \in \{0, 1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

German: allgemeines Halteproblem, Halteproblem

General Halting Problem (1)

Definition (General Halting Problem)

The **general halting problem** or **halting problem** is the language

$$H = \{w\#x \in \{0, 1, \#\}^* \mid w, x \in \{0, 1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

German: allgemeines Halteproblem, Halteproblem

Note: H is semi-decidable. ([Why?](#))

General Halting Problem (1)

Definition (General Halting Problem)

The **general halting problem** or **halting problem** is the language

$$H = \{w\#x \in \{0, 1, \#\}^* \mid w, x \in \{0, 1\}^*, \\ M_w \text{ started on } x \text{ terminates}\}$$

German: allgemeines Halteproblem, Halteproblem

Note: H is semi-decidable. ([Why?](#))

Theorem (Undecidability of General Halting Problem)

The general halting problem is undecidable.

Intuition: if the special case K is not decidable, then the more general problem H definitely cannot be decidable.

General Halting Problem (2)

Proof.

We show $K \leq H$.

We define $f : \{0, 1\}^* \rightarrow \{0, 1, \#\}^*$ as $f(w) := w\#w$.

f is clearly total and computable, and

$$w \in K$$

iff M_w started on w terminates

iff $w\#w \in H$

iff $f(w) \in H$.

General Halting Problem (2)

Proof.

We show $K \leq H$.

We define $f : \{0, 1\}^* \rightarrow \{0, 1, \#\}^*$ as $f(w) := w\#w$.

f is clearly total and computable, and

$$w \in K$$

iff M_w started on w terminates

iff $w\#w \in H$

iff $f(w) \in H$.

Therefore f is a reduction from K to H .

Because K is undecidable, H is also undecidable. □

Halting Problem on Empty Tape (1)

Definition (Halting Problem on the Empty Tape)

The **halting problem on the empty tape** is the language

$$H_0 = \{w \in \{0, 1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

German: Halteproblem auf leerem Band

Halting Problem on Empty Tape (1)

Definition (Halting Problem on the Empty Tape)

The **halting problem on the empty tape** is the language

$$H_0 = \{w \in \{0, 1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

German: Halteproblem auf leerem Band

Note: H_0 is semi-decidable. ([Why?](#))

Halting Problem on Empty Tape (1)

Definition (Halting Problem on the Empty Tape)

The **halting problem on the empty tape** is the language

$$H_0 = \{w \in \{0, 1\}^* \mid M_w \text{ started on } \varepsilon \text{ terminates}\}.$$

German: Halteproblem auf leerem Band

Note: H_0 is semi-decidable. ([Why?](#))

Theorem (Undecidability of Halting Problem on Empty Tape)

The halting problem on the empty tape is undecidable.

Halting Problem on Empty Tape (2)

Proof.

We show $H \leq H_0$.

Halting Problem on Empty Tape (2)

Proof.

We show $H \leq H_0$.

Consider the function $f : \{0, 1, \#\}^* \rightarrow \{0, 1\}^*$

that computes the word $f(z)$ for a given $z \in \{0, 1, \#\}^*$ as follows:

Halting Problem on Empty Tape (2)

Proof.

We show $H \leq H_0$.

Consider the function $f : \{0, 1, \#\}^* \rightarrow \{0, 1\}^*$

that computes the word $f(z)$ for a given $z \in \{0, 1, \#\}^*$ as follows:

- Test if z has the form $w\#x$ with $w, x \in \{0, 1\}^*$.
- If not, return any word that is not in H_0
(e.g., encoding of a TM that instantly starts an endless loop).
- If yes, split z into w and x .

Halting Problem on Empty Tape (2)

Proof.

We show $H \leq H_0$.

Consider the function $f : \{0, 1, \#\}^* \rightarrow \{0, 1\}^*$

that computes the word $f(z)$ for a given $z \in \{0, 1, \#\}^*$ as follows:

- Test if z has the form $w\#x$ with $w, x \in \{0, 1\}^*$.
- If not, return any word that is not in H_0
(e.g., encoding of a TM that instantly starts an endless loop).
- If yes, split z into w and x .
- Decode w to a TM M_2 .

...

Halting Problem on Empty Tape (3)

Proof (continued).

- Construct a TM M_1 that behaves as follows:
 - If the input is empty: write x onto the tape and move the head to the first symbol of x (if $x \neq \varepsilon$); then stop
 - otherwise, stop immediately

Halting Problem on Empty Tape (3)

Proof (continued).

- Construct a TM M_1 that behaves as follows:
 - If the input is empty: write x onto the tape and move the head to the first symbol of x (if $x \neq \varepsilon$); then stop
 - otherwise, stop immediately
- Construct TM M that first runs M_1 and then M_2 .

Halting Problem on Empty Tape (3)

Proof (continued).

- Construct a TM M_1 that behaves as follows:
 - If the input is empty: write x onto the tape and move the head to the first symbol of x (if $x \neq \varepsilon$); then stop
 - otherwise, stop immediately
- Construct TM M that first runs M_1 and then M_2 .
- Return the encoding of M .

Halting Problem on Empty Tape (3)

Proof (continued).

- Construct a TM M_1 that behaves as follows:
 - If the input is empty: write x onto the tape and move the head to the first symbol of x (if $x \neq \varepsilon$); then stop
 - otherwise, stop immediately
- Construct TM M that first runs M_1 and then M_2 .
- Return the encoding of M .

f is total and (with some effort) computable. Also:

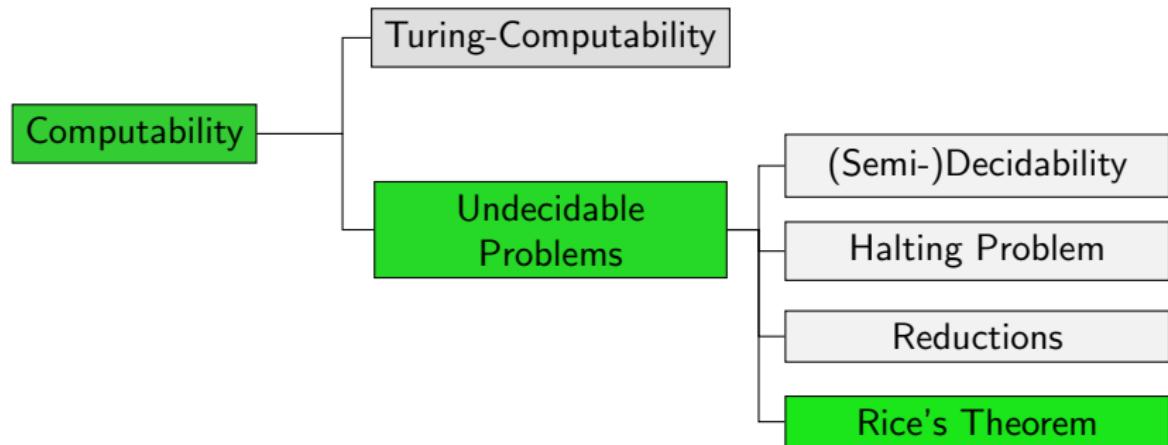
$z \in H$ iff $z = w\#x$ and M_w run on x terminates
iff $M_{f(z)}$ started on empty tape terminates
iff $f(z) \in H_0$

$\rightsquigarrow H \leq H_0 \rightsquigarrow H_0$ undecidable



Rice's Theorem

Overview: Computability Theory



Rice's Theorem (1)

- We have shown that a number of (related) problems are undecidable:
 - special halting problem K
 - general halting problem H
 - halting problem on empty tape H_0
- Many more results of this type could be shown.
- Instead, we prove a much more general result, **Rice's theorem**, which shows that a very large class of different problems are undecidable.
- Rice's theorem can be summarized informally as: **every** non-trivial question about **what** a given Turing machine computes is undecidable.

Rice's Theorem (2)

Theorem (Rice's Theorem)

Let \mathcal{R} be the class of all computable functions.

Let \mathcal{S} be an **arbitrary** subset of \mathcal{R} except $\mathcal{S} = \emptyset$ or $\mathcal{S} = \mathcal{R}$.

Then the language

$$C(\mathcal{S}) = \{w \in \{0, 1\}^* \mid \text{the function computed by } M_w \text{ is in } \mathcal{S}\}$$

is undecidable.

German: Satz von Rice

Question: why the restriction to $\mathcal{S} \neq \emptyset$ and $\mathcal{S} \neq \mathcal{R}$?

Extension (without proof): in most cases neither $C(\mathcal{S})$ nor $\overline{C(\mathcal{S})}$ is semi-decidable. (But there are sets \mathcal{S} for which one of the two languages is semi-decidable.)

Rice's Theorem (3)

Proof.

Let Ω be the function that is undefined everywhere.

Rice's Theorem (3)

Proof.

Let Ω be the function that is undefined everywhere.

Case distinction:

Case 1: $\Omega \in \mathcal{S}$

Rice's Theorem (3)

Proof.

Let Ω be the function that is undefined everywhere.

Case distinction:

Case 1: $\Omega \in \mathcal{S}$

Let $q \in \mathcal{R} \setminus \mathcal{S}$ be an arbitrary computable function outside of \mathcal{S} (exists because $\mathcal{S} \subseteq \mathcal{R}$ and $\mathcal{S} \neq \mathcal{R}$).

Rice's Theorem (3)

Proof.

Let Ω be the function that is undefined everywhere.

Case distinction:

Case 1: $\Omega \in \mathcal{S}$

Let $q \in \mathcal{R} \setminus \mathcal{S}$ be an arbitrary computable function outside of \mathcal{S} (exists because $\mathcal{S} \subseteq \mathcal{R}$ and $\mathcal{S} \neq \mathcal{R}$).

Let Q be a Turing machine that computes q .

...

Rice's Theorem (4)

Proof (continued).

We show that $\bar{H}_0 \leq C(\mathcal{S})$.

Consider function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$,
where $f(w)$ is defined as follows:

- Construct TM M that first behaves on input y like M_w on the empty tape (independently of what y is).
- Afterwards (if that computation terminates!)
 M clears the tape, creates the start configuration of Q for input y and then simulates Q .
- $f(w)$ is the encoding of this TM M

Rice's Theorem (4)

Proof (continued).

We show that $\bar{H}_0 \leq C(\mathcal{S})$.

Consider function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$,
where $f(w)$ is defined as follows:

- Construct TM M that first behaves on input y like M_w on the empty tape (independently of what y is).
- Afterwards (if that computation terminates!)
 M clears the tape, creates the start configuration of Q for input y and then simulates Q .
- $f(w)$ is the encoding of this TM M

f is total and computable.

...

Rice's Theorem (5)

Proof (continued).

Which function is computed by the TM encoded by $f(w)$?

Rice's Theorem (5)

Proof (continued).

Which function is computed by the TM encoded by $f(w)$?

$M_{f(w)}$ computes $\begin{cases} \Omega & \text{if } M_w \text{ does not terminate on } \varepsilon \\ q & \text{otherwise} \end{cases}$

Rice's Theorem (5)

Proof (continued).

Which function is computed by the TM encoded by $f(w)$?

$M_{f(w)}$ computes $\begin{cases} \Omega & \text{if } M_w \text{ does not terminate on } \varepsilon \\ q & \text{otherwise} \end{cases}$

For all words $w \in \{0, 1\}^*$:

$w \in H_0 \implies M_w \text{ terminates on } \varepsilon$

$\implies M_{f(w)}$ computes the function q

\implies the function computed by $M_{f(w)}$ is not in \mathcal{S}

$\implies f(w) \notin C(\mathcal{S})$

...

Rice's Theorem (6)

Proof (continued).

Further:

- $w \notin H_0 \implies M_w \text{ does not terminate on } \varepsilon$
- $\implies M_{f(w)} \text{ computes the function } \Omega$
- $\implies \text{the function computed by } M_{f(w)} \text{ is in } \mathcal{S}$
- $\implies f(w) \in C(\mathcal{S})$

Rice's Theorem (6)

Proof (continued).

Further:

- $w \notin H_0 \implies M_w \text{ does not terminate on } \varepsilon$
- $\implies M_{f(w)} \text{ computes the function } \Omega$
- $\implies \text{the function computed by } M_{f(w)} \text{ is in } \mathcal{S}$
- $\implies f(w) \in C(\mathcal{S})$

Together this means: $w \notin H_0$ iff $f(w) \in C(\mathcal{S})$,
thus $w \in \bar{H}_0$ iff $f(w) \in C(\mathcal{S})$.

Rice's Theorem (6)

Proof (continued).

Further:

- $w \notin H_0 \implies M_w \text{ does not terminate on } \varepsilon$
- $\implies M_{f(w)} \text{ computes the function } \Omega$
- $\implies \text{the function computed by } M_{f(w)} \text{ is in } \mathcal{S}$
- $\implies f(w) \in C(\mathcal{S})$

Together this means: $w \notin H_0$ iff $f(w) \in C(\mathcal{S})$,
thus $w \in \bar{H}_0$ iff $f(w) \in C(\mathcal{S})$.

Therefore, f is a reduction of \bar{H}_0 to $C(\mathcal{S})$.

Rice's Theorem (6)

Proof (continued).

Further:

- $w \notin H_0 \implies M_w \text{ does not terminate on } \varepsilon$
- $\implies M_{f(w)} \text{ computes the function } \Omega$
- $\implies \text{the function computed by } M_{f(w)} \text{ is in } \mathcal{S}$
- $\implies f(w) \in C(\mathcal{S})$

Together this means: $w \notin H_0$ iff $f(w) \in C(\mathcal{S})$,
thus $w \in \bar{H}_0$ iff $f(w) \in C(\mathcal{S})$.

Therefore, f is a reduction of \bar{H}_0 to $C(\mathcal{S})$.

Since H_0 is undecidable, \bar{H}_0 is also undecidable.

Rice's Theorem (6)

Proof (continued).

Further:

- $w \notin H_0 \implies M_w \text{ does not terminate on } \varepsilon$
- $\implies M_{f(w)} \text{ computes the function } \Omega$
- $\implies \text{the function computed by } M_{f(w)} \text{ is in } \mathcal{S}$
- $\implies f(w) \in C(\mathcal{S})$

Together this means: $w \notin H_0$ iff $f(w) \in C(\mathcal{S})$,
thus $w \in \bar{H}_0$ iff $f(w) \in C(\mathcal{S})$.

Therefore, f is a reduction of \bar{H}_0 to $C(\mathcal{S})$.

Since H_0 is undecidable, \bar{H}_0 is also undecidable.

We can conclude that $C(\mathcal{S})$ is undecidable.

...

Rice's Theorem (7)

Proof (continued).

Case 2: $\Omega \notin \mathcal{S}$

Analogous to Case 1 but this time choose $q \in \mathcal{S}$.

The corresponding function f then reduces H_0 to $C(\mathcal{S})$.

Thus, it also follows in this case that $C(\mathcal{S})$ is undecidable. □

Rice's Theorem: Consequences

Was it worth it?

We can now conclude immediately that (for example) the following informally specified problems are all undecidable:

- Does a given TM compute a constant function?
- Does a given TM compute a total function
(i. e. will it always terminate, and in particular terminate in a “correct” configuration)?
- Is the output of a given TM always longer than its input?
- Does a given TM compute the identity function?
- Does a given TM compute the computable function f ?
- ...

Rice's Theorem: Examples

- Does a given TM compute a constant function?

$\mathcal{S} = \{f \mid f \text{ is total and computable and}$
 $\text{for all } x, y \text{ in the domain of } f : f(x) = f(y)\}$

- Does a given TM compute a total function?

$\mathcal{S} = \{f \mid f \text{ is total and computable}\}$

- Does a given TM compute the identity function?

$\mathcal{S} = \{f \mid f(x) = x \text{ for all } x\}$

- Does a given TM add two natural numbers?

$\mathcal{S} = \{f : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 \mid f(x, y) = x + y\}$

- Does a given TM compute the computable function f ?

$\mathcal{S} = \{f\}$

(full automation of software verification is impossible)

Rice's Theorem: Pitfalls

- $\mathcal{S} = \{f \mid f \text{ can be computed by a DTM with an even number of states}\}$
Rice's theorem not applicable because $\mathcal{S} = \mathcal{R}$
- $\mathcal{S} = \{f : \{0, 1\}^* \rightarrow_p \{0, 1\} \mid f(w) = 1 \text{ iff } M_w \text{ does not terminate on } \epsilon\}$?
Rice's theorem not applicable because $\mathcal{S} \not\subseteq \mathcal{R}$
- Show that $\{w \mid M_w \text{ traverses all states on every input}\}$ is undecidable.
Rice's theorem not directly applicable because not a semantic property (the function computed by M_w can also be computed by a TM that does not traverse all states)

Rice's Theorem: Practical Applications

Undecidable due to Rice's theorem + a small reduction:

- **automated debugging:**
 - Can a given variable ever receive a `null` value?
 - Can a given assertion in a program ever trigger?
 - Can a given buffer ever overflow?
- **virus scanners and other software security analysis:**
 - Can this code do something harmful?
 - Is this program vulnerable to SQL injections?
 - Can this program lead to a privilege escalation?
- **optimizing compilers:**
 - Is this dead code?
 - Is this a constant expression?
 - Can pointer aliasing happen here?
 - Is it safe to parallelize this code path?
- **parallel program analysis:**
 - Is a deadlock possible here?
 - Can a race condition happen here?

Summary

Summary

undecidable but semi-decidable problems:

- **special halting problem** a.k.a. self-application problem
(from previous chapter)
- **general halting problem**
- **halting problem on empty tape**

Rice's theorem:

- “In general one cannot determine algorithmically what a given program (or Turing machine) computes.”