

Theory of Computer Science

D3. Halting Problem and Reductions

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Introduction

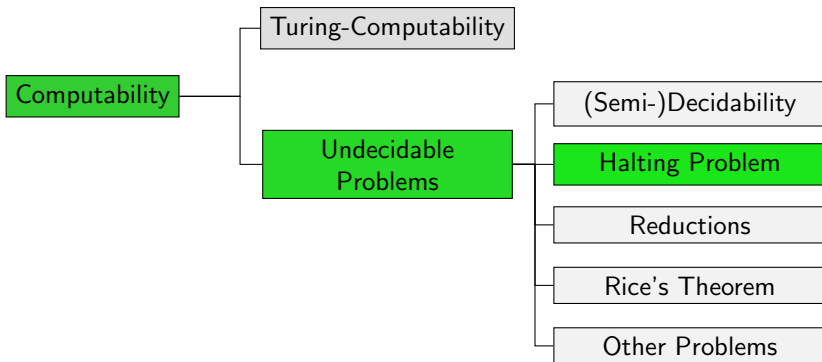
Undecidable Problems

- We now know many characterizations of semi-decidability and decidability.
- What's missing is a **concrete example** for an **undecidable** (= not decidable) problem.
- Do undecidable problems even exist?

Undecidable Problems

- We now know many characterizations of semi-decidability and decidability.
- What's missing is a **concrete example** for an **undecidable** (= not decidable) problem.
- Do undecidable problems even exist?
- Yes! **Counting argument**: there are (for a fixed Σ) as many **decision algorithms** (e. g., Turing machines) as numbers in \mathbb{N}_0 but as many **languages** as numbers in \mathbb{R} .
Since \mathbb{N}_0 cannot be surjectively mapped to \mathbb{R} , languages with no decision algorithm exist.
- But this argument does not give us a **concrete** undecidable problem. \rightsquigarrow main goal of this chapter

Overview: Computability Theory



Turing Machines as Words

Turing Machines as Inputs

- The first undecidable problems that we will get to know have Turing machines as their **input**.
↪ “programs that have programs as input”:
cf. compilers, interpreters, virtual machines, etc.
- We have to think about how we can encode **arbitrary Turing machines** as **words over a fixed alphabet**.
- We use the binary alphabet $\Sigma = \{0, 1\}$.
- As an intermediate step we first encode over the alphabet $\Sigma' = \{0, 1, \#\}$.

Encoding a Turing Machine as a Word (1)

Step 1: encode a Turing machine as a word over $\{0, 1, \#\}$

Reminder: Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$

Idea:

- input alphabet Σ should always be $\{0, 1\}$
- enumerate states in Q and symbols in Γ and consider them as numbers $0, 1, 2, \dots$
- blank symbol always receives number 2
- start state always receives number 0

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Then it is sufficient to **only encode δ** explicitly:

- Q : all states mentioned in the encoding of δ
- E : all states that never occur on a left-hand side of a δ -rule
- $\Gamma = \{0, 1, \square, a_3, a_4, \dots, a_k\}$, where k is the largest symbol number mentioned in the δ -rules

Encoding a Turing Machine as a Word (2)

encode the rules:

- Let $\delta(q_i, a_j) = \langle q_{i'}, a_{j'}, D \rangle$ be a rule in δ , where the indices i, i', j, j' correspond to the enumeration of states/symbols and $D \in \{L, R, N\}$.

- encode this rule as

$$w_{i,j,i',j',D} = \#\#bin(i)\#bin(j)\#bin(i')\#bin(j')\#bin(m),$$

$$\text{where } m = \begin{cases} 0 & \text{if } D = L \\ 1 & \text{if } D = R \\ 2 & \text{if } D = N \end{cases}$$

- For every rule in δ , we obtain one such word.
- All of these words in sequence (in arbitrary order) encode the Turing machine.

Encoding a Turing Machine as a Word (3)

Step 2: transform into word over $\{0, 1\}$ with mapping

$0 \mapsto 00$

$1 \mapsto 01$

$\# \mapsto 11$

Turing machine can be reconstructed from its encoding.

How?

Encoding a Turing Machine as a Word (4)

Example (step 1)

$\delta(q_2, a_3) = \langle q_0, a_2, N \rangle$ becomes ##10#11#0#10#10

$\delta(q_1, a_1) = \langle q_3, a_0, L \rangle$ becomes ##1#1#11#0#0

Example (step 2)

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Note: We can also consider the encoded word (uniquely; [why?](#)) as a **number** that enumerates this TM.

This is not important for the halting problem but in other contexts where we operate on numbers instead of words.

Turing Machine Encoded by a Word

goal: function that maps any word in $\{0, 1\}^*$ to a Turing machine

problem: not all words in $\{0, 1\}^*$ are encodings of a Turing machine

solution: Let \hat{M} be an arbitrary fixed deterministic Turing machine (for example one that always immediately stops). Then:

Definition (Turing Machine Encoded by a Word)

For all $w \in \{0, 1\}^*$:

$$M_w = \begin{cases} M' & \text{if } w \text{ is the encoding of some DTM } M' \\ \hat{M} & \text{otherwise} \end{cases}$$

Special Halting Problem

Special Halting Problem

Our preparations are now done and we can define:

Definition (Special Halting Problem)

The **special halting problem** or **self-application problem** is the language

$$K = \{w \in \{0, 1\}^* \mid M_w \text{ started on } w \text{ terminates}\}.$$

German: spezielles Halteproblem, Selbstanwendbarkeitsproblem

Note: word w plays two roles as encoding of the TM and as input for encoded machine

Semi-Decidability of the Special Halting Problem

Theorem (Semi-Decidability of the Special Halting Problem)

The special halting problem is semi-decidable.

Proof.

We construct an “interpreter” for DTMs that receives the encoding of a DTM as input w and simulates its computation on input w .

If the simulated DTM stops, the interpreter returns 1.

Otherwise it does not return.

This interpreter computes χ'_K . □

Note: TMs simulating arbitrary TMs are called **universal** TMs.

German: universelle Turingmaschine

Undecidability of the Special Halting Problem (1)

Theorem (Undecidability of the Special Halting Problem)

The special halting problem is undecidable.

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Proof by contradiction: we assume that the special halting problem K were decidable and derive a contradiction.

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So assume K is decidable. Then χ_K is computable (**why?**).

Undecidability of the Special Halting Problem (1)

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The special halting problem is undecidable.

Proof.

Proof by contradiction: we assume that the special halting problem K were decidable and derive a contradiction.

So assume K is decidable. Then χ_K is computable (**why?**).

Let M be a Turing machine that computes χ_K , i. e., given a word w writes 1 or 0 onto the tape (depending on whether $w \in K$) and then stops. ...

Undecidability of the Special Halting Problem (2)

Proof (continued).

Construct a new machine M' as follows:

- 1 Execute M on the input w .
- 2 If the tape content is 0: stop.
- 3 Otherwise: enter an endless loop.

Undecidability of the Special Halting Problem (2)

Proof (continued).

Construct a new machine M' as follows:

- 1 Execute M on the input w .
- 2 If the tape content is 0: stop.
- 3 Otherwise: enter an endless loop.

Let w' be the encoding of M' . **How will M' behave on input w' ?**

Undecidability of the Special Halting Problem (2)

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M' run on w' stops

iff M run on w outputs 0

Undecidability of the Special Halting Problem (2)

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iff M run on w outputs 0

iff $\chi_K(w') = 0$

Undecidability of the Special Halting Problem (2)

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- ① Execute M on the input w .
- ② If the tape content is 0: stop.
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iff $w' \notin K$

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M' run on w' stops

iff M run on w outputs 0

iff $\chi_K(w) = 0$

iff $w \notin K$

iff M_w run on w does not stop

Undecidability of the Special Halting Problem (2)

Proof (continued).

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- 1 Execute M on the input w .
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Let w' be the encoding of M' . **How will M' behave on input w' ?**

M' run on w' stops

iff M run on w' outputs 0

iff $\chi_K(w') = 0$

iff $w' \notin K$

iff $M_{w'}$ run on w' does not stop

iff M' run on w' does not stop

Contradiction! This proves the theorem.



Reprise: Type-0 Languages

Back to Chapter C8: Closure Properties

	Intersection	Union	Complement	Concatenation	Star
Type 3	Yes	Yes	Yes	Yes	Yes
Type 2	No	Yes	No	Yes	Yes
Type 1	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽²⁾	Yes ⁽¹⁾	Yes ⁽¹⁾
Type 0	Yes ⁽²⁾	Yes ⁽¹⁾	No ⁽³⁾	Yes ⁽¹⁾	Yes ⁽¹⁾

Proofs?

(1) proof via grammars, similar to context-free cases

(2) without proof

(3) proof in later chapters (part D)

Back to Chapter C8: Decidability

	Word problem	Emptiness problem	Equivalence problem	Intersection problem
Type 3	Yes	Yes	Yes	Yes
Type 2	Yes	Yes	No	No
Type 1	Yes ⁽¹⁾	No ⁽³⁾	No ⁽²⁾	No ⁽²⁾
Type 0	No ⁽⁴⁾	No ⁽⁴⁾	No ⁽⁴⁾	No ⁽⁴⁾

Proofs?

- (1) same argument we used for context-free languages
- (2) because already undecidable for context-free languages
- (3) without proof
- (4) proofs in later chapters (part D)

Answers to Old Questions

Closure properties:

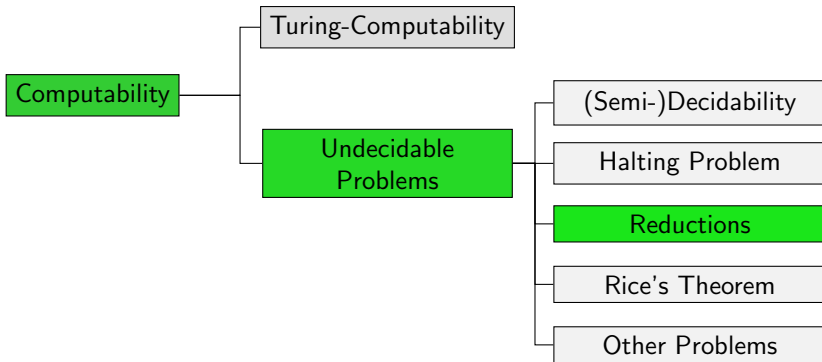
- K is semi-decidable (and thus type 0) but not decidable.
- ↪ \bar{K} is **not** semi-decidable, thus **not** type 0.
- ↪ Type-0 languages are **not** closed under complement.

Decidability:

- K is type 0 but not decidable.
- ↪ **word problem** for type-0 languages not decidable
- ↪ emptiness, equivalence, intersection problem: **later in exercises**
(We are still missing some important results for this.)

Reductions

Overview: Computability Theory



What We Achieved So Far: Discussion

- We now know a concrete undecidable problem.
- But the problem is rather artificial:
how often do we want to apply a program to itself?
- We will see that we can derive **further** (more useful) undecidability results from the undecidability of the special halting problem.
- The central notion for this is **reducing** a new problem to an already known problem.

Reductions: Definition

Definition (Reduction)

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be languages, and let $f : \Sigma^* \rightarrow \Gamma^*$ be a total and computable function such that for all $x \in \Sigma^*$:

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

Then we say that A can be **reduced to B** (in symbols: $A \leq B$), and f is called **reduction from A to B** .

German: A ist auf B reduzierbar, Reduktion von A auf B

Reduction Property

Theorem (Reductions vs. Semi-Decidability/Decidability)

Let A and B be languages with $A \leq B$. Then:

- 1 If B is decidable, then A is decidable.
- 2 If B is semi-decidable, then A is semi-decidable.
- 3 If A is not decidable, then B is not decidable.
- 4 If A is not semi-decidable, then B is not semi-decidable.

↪ In the following, we use 3. to show undecidability for further problems.

Reduction Property: Proof

Proof.

for 1.: The following algorithm computes $\chi_A(x)$ given input x :

$y := f(x)$

result := $\chi_B(y)$

RETURN result

Reduction Property: Proof

Proof.

for 1.: The following algorithm computes $\chi_A(x)$ given input x :

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for 2.: identical to (1), but use χ'_B (instead of χ_B)

to compute χ'_A (instead of χ_A)

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for 1.: The following algorithm computes $\chi_A(x)$ given input x :

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RETURN result

for 2.: identical to (1), but use χ'_B (instead of χ_B)

to compute χ'_A (instead of χ_A)

for 3./4.: contrapositions of 1./2. \rightsquigarrow logically equivalent



Reductions are Preorders

Theorem (Reductions are Preorders)

The relation " \leq " is a preorder:

- 1 For all languages A :
 $A \leq A$ (*reflexivity*)
- 2 For all languages A, B, C :
If $A \leq B$ and $B \leq C$, then $A \leq C$ (*transitivity*)

German: schwache Halbordnung/Quasiordnung, Reflexivität, Transitivität

Reductions are Preorders: Proof

Proof.

for 1.: The function $f(x) = x$ is a reduction from A to A because it is total and computable and $x \in A$ iff $f(x) \in A$.

for 2.: \rightsquigarrow exercises



Summary

Summary

- The **special halting problem** (self-application problem) is undecidable.
- However, it is semi-decidable.
- important concept in this chapter:
Turing machines represented as **words**
↪ Turing machines taking Turing machines as their input
- **reductions**: “embedding” a problem as a special case of another problem
- important method for proving undecidability:
reduce from a known undecidable problem to a new problem