# Theory of Computer Science

## D3. Halting Problem and Reductions

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# D3.1 Introduction



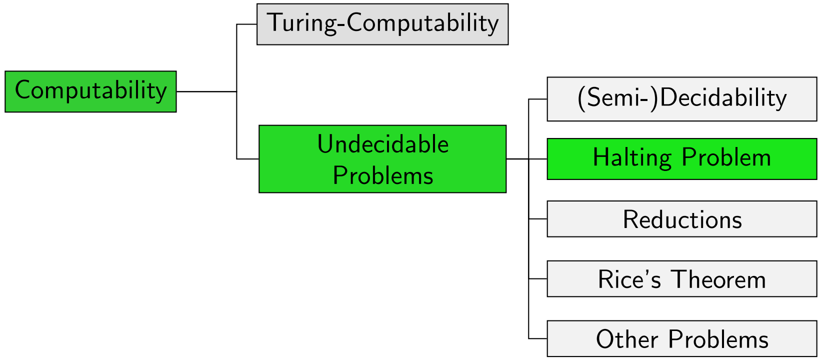


## Undecidable Problems

- We now know many characterizations of semi-decidability and decidability.
- What's missing is a concrete example for an undecidable (= not decidable) problem.
- Do undecidable problems even exist?
- Yes! Counting argument: there are (for a fixed $\Sigma$) as many decision algorithms (e. g., Turing machines) as numbers in $\mathbb{N}_0$ but as many languages as numbers in $\mathbb{R}$.

  Since $\mathbb{N}_0$ cannot be surjectively mapped to $\mathbb{R}$, languages with no decision algorithm exist.
- But this argument does not give us a concrete undecidable problem. $\leadsto$ main goal of this chapter

## Overview: Computability Theory








# D3.2 Turing Machines as Words





## Turing Machines as Inputs

- The first undecidable problems that we will get to know have Turing machines as their input.
  $\rightsquigarrow$ "programs that have programs as input":
  cf. compilers, interpreters, virtual machines, etc.
- We have to think about how we can encode arbitrary Turing machines as words over a fixed alphabet.
- We use the binary alphabet $\Sigma = \{0, 1\}$.
- As an intermediate step we first encode over the alphabet $\Sigma' = \{0, 1, \#\}$.





## Encoding a Turing Machine as a Word (1)

Step 1: encode a Turing machine as a word over $\{0, 1, \#\}$

Reminder: Turing machine $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$

Idea:

- input alphabet $\Sigma$ should always be $\{0, 1\}$
- enumerate states in $Q$ and symbols in $\Gamma$ and consider them as numbers $0, 1, 2, \dots$
- blank symbol always receives number 2
- start state always receives number 0

Then it is sufficient to only encode $\delta$ explicitly:

- $Q$: all states mentioned in the encoding of $\delta$
- $E$: all states that never occur on a left-hand side of a $\delta$-rule
- $\Gamma = \{0, 1, \square, a_3, a_4, \dots, a_k\}$, where $k$ is the largest symbol number mentioned in the $\delta$-rules

## Encoding a Turing Machine as a Word (2)

encode the rules:

- Let $\delta(q_i, a_j) = \langle q_{i'}, a_{j'}, D\rangle$ be a rule in $\delta$, where the indices $i$, $i'$, $j$, $j'$ correspond to the enumeration of states/symbols and $D \in \{\mathrm{L}, \mathrm{R}, \mathrm{N}\}$.
- encode this rule as $w_{i,j,i',j',D} = \#\#bin(i)\#bin(j)\#bin(i')\#bin(j')\#bin(m)$, where $m = \begin{cases} 0 & \text{if } D = \mathrm{L} \\ 1 & \text{if } D = \mathrm{R} \\ 2 & \text{if } D = \mathrm{N} \end{cases}$
- For every rule in $\delta$, we obtain one such word.
- All of these words in sequence (in arbitrary order) encode the Turing machine.





## Encoding a Turing Machine as a Word (3)

Step 2: transform into word over $\{0, 1\}$ with mapping

$$0 \mapsto 00$$
$$1 \mapsto 01$$
$$\# \mapsto 11$$

Turing machine can be reconstructed from its encoding.
How?





## Encoding a Turing Machine as a Word (4)

**Example (step 1)**

$\delta(q_2, a_3) = \langle q_0, a_2, N\rangle$ becomes ##10#11#0#10#10
$\delta(q_1, a_1) = \langle q_3, a_0, L\rangle$ becomes ##1#1#11#0#0

**Example (step 2)**

##10#11#0#10#10##1#1#11#0#0
111101001101011100110100110100111101110111010111001100

Note: We can also consider the encoded word (uniquely; why?) as a number that enumerates this TM.

This is not important for the halting problem but in other contexts where we operate on numbers instead of words.





## Turing Machine Encoded by a Word

goal: function that maps any word in $\{0, 1\}^*$ to a Turing machine
problem: not all words in $\{0, 1\}^*$ are encodings of a Turing machine

solution: Let $\widehat{M}$ be an arbitrary fixed deterministic Turing machine (for example one that always immediately stops). Then:

**Definition (Turing Machine Encoded by a Word)**

For all $w \in \{0, 1\}^*$:

$$M_w = \begin{cases} M' & \text{if } w \text{ is the encoding of some DTM } M' \\ \widehat{M} & \text{otherwise} \end{cases}$$

# D3.3 Special Halting Problem





## Special Halting Problem

Our preparations are now done and we can define:

**Definition (Special Halting Problem)**

The special halting problem or self-application problem is the language

$$K = \{w \in \{0,1\}^* \mid M_w \text{ started on } w \text{ terminates}\}.$$

German: spezielles Halteproblem, Selbstanwendbarkeitsproblem

Note: word $w$ plays two roles as encoding of the TM and as input for encoded machine





## Semi-Decidability of the Special Halting Problem

**Theorem (Semi-Decidability of the Special Halting Problem)**

*The special halting problem is semi-decidable.*

**Proof.**

We construct an "interpreter" for DTMs that receives the encoding of a DTM as input $w$ and simulates its computation on input $w$.

If the simulated DTM stops, the interpreter returns 1. Otherwise it does not return.

This interpreter computes $\chi'_K$. □

Note: TMs simulating arbitrary TMs are called universal TMs.

German: universelle Turingmaschine





## Undecidability of the Special Halting Problem (1)

**Theorem (Undecidability of the Special Halting Problem)**

*The special halting problem is undecidable.*

**Proof.**

Proof by contradiction: we assume that the special halting problem $K$ were decidable and derive a contradiction.

So assume $K$ is decidable. Then $\chi_K$ is computable (why?).

Let $M$ be a Turing machine that computes $\chi_K$, i. e., given a word $w$ writes 1 or 0 onto the tape (depending on whether $w \in K$) and then stops. ...

## Undecidability of the Special Halting Problem (2)

Proof (continued).

Construct a new machine $M'$ as follows:

1. Execute $M$ on the input $w$.
2. If the tape content is 0: stop.
3. Otherwise: enter an endless loop.

Let $w'$ be the encoding of $M'$. How will $M'$ behave on input $w'$?

$M'$ run on $w'$ stops
iff $M$ run on $w'$ outputs 0
iff $\chi_K(w') = 0$
iff $w' \notin K$
iff $M_{w'}$ run on $w'$ does not stop
iff $M'$ run on $w'$ does not stop

Contradiction! This proves the theorem. □





# D3.4 Reprise: Type-0 Languages





## Back to Chapter C8: Closure Properties

| | Intersection | Union | Complement | Concatenation | Star |
|---|---|---|---|---|---|
| Type 3 | Yes | Yes | Yes | Yes | Yes |
| Type 2 | No | Yes | No | Yes | Yes |
| Type 1 | Yes$^{(2)}$ | Yes$^{(1)}$ | Yes$^{(2)}$ | Yes$^{(1)}$ | Yes$^{(1)}$ |
| Type 0 | Yes$^{(2)}$ | Yes$^{(1)}$ | No$^{(3)}$ | Yes$^{(1)}$ | Yes$^{(1)}$ |

Proofs?
(1) proof via grammars, similar to context-free cases
(2) without proof
(3) proof in later chapters (part D)





## Back to Chapter C8: Decidability

| | Word problem | Emptiness problem | Equivalence problem | Intersection problem |
|---|---|---|---|---|
| Type 3 | Yes | Yes | Yes | Yes |
| Type 2 | Yes | Yes | No | No |
| Type 1 | Yes$^{(1)}$ | No$^{(3)}$ | No$^{(2)}$ | No$^{(2)}$ |
| Type 0 | No$^{(4)}$ | No$^{(4)}$ | No$^{(4)}$ | No$^{(4)}$ |

Proofs?
(1) same argument we used for context-free languages
(2) because already undecidable for context-free languages
(3) without proof
(4) proofs in later chapters (part D)

## Answers to Old Questions

Closure properties:

- $K$ is semi-decidable (and thus type 0) but not decidable.
- ⇝ $\bar{K}$ is not semi-decidable, thus not type 0.
- ⇝ Type-0 languages are not closed under complement.

Decidability:

- $K$ is type 0 but not decidable.
- ⇝ word problem for type-0 languages not decidable
- ⇝ emptiness, equivalence, intersection problem: later in exercises
  (We are still missing some important results for this.)



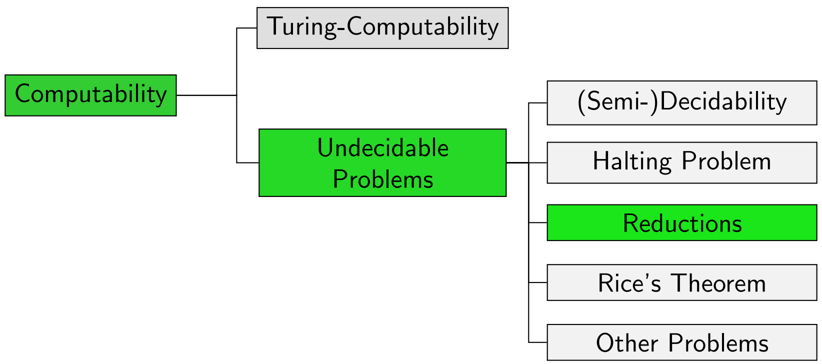


# D3.5 Reductions





## Overview: Computability Theory








## What We Achieved So Far: Discussion

- We now know a concrete undecidable problem.
- But the problem is rather artificial:
  how often do we want to apply a program to itself?
- We will see that we can derive further (more useful) undecidability results from the undecidability of the special halting problem.
- The central notion for this is reducing a new problem to an already known problem.

## Reductions: Definition

**Definition (Reduction)**

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be languages, and let $f : \Sigma^* \to \Gamma^*$ be a total and computable function such that for all $x \in \Sigma^*$:

$$x \in A \quad \text{if and only if} \quad f(x) \in B.$$

Then we say that $A$ can be reduced to $B$ (in symbols: $A \leq B$), and $f$ is called reduction from $A$ to $B$.

German: $A$ ist auf $B$ reduzierbar, Reduktion von $A$ auf $B$





## Reduction Property

**Theorem (Reductions vs. Semi-Decidability/Decidability)**

*Let $A$ and $B$ be languages with $A \leq B$. Then:*

1. *If $B$ is decidable, then $A$ is decidable.*
2. *If $B$ is semi-decidable, then $A$ is semi-decidable.*
3. *If $A$ is not decidable, then $B$ is not decidable.*
4. *If $A$ is not semi-decidable, then $B$ is not semi-decidable.*

$\rightsquigarrow$ In the following, we use 3. to show undecidability for further problems.





## Reduction Property: Proof

**Proof.**

for 1.: The following algorithm computes $\chi_A(x)$ given input $x$:

$y := f(x)$
result $:= \chi_B(y)$
RETURN result

for 2.: identical to (1), but use $\chi'_B$ (instead of $\chi_B$) to compute $\chi'_A$ (instead of $\chi_A$)

for 3./4.: contrapositions of 1./2. $\rightsquigarrow$ logically equivalent □





## Reductions are Preorders

**Theorem (Reductions are Preorders)**

*The relation "$\leq$" is a preorder:*

1. *For all languages $A$:*
   *$A \leq A$ (reflexivity)*
2. *For all languages $A, B, C$:*
   *If $A \leq B$ and $B \leq C$, then $A \leq C$ (transitivity)*

German: schwache Halbordnung/Quasiordnung, Reflexivität, Transitivität

## Reductions are Preorders: Proof

Proof.
for 1.: The function $f(x) = x$ is a reduction from $A$ to $A$ because it is total and computable and $x \in A$ iff $f(x) \in A$.

for 2.: $\rightsquigarrow$ exercises $\square$





# D3.6 Summary





## Summary

- The special halting problem (self-application problem) is undecidable.
- However, it is semi-decidable.
- important concept in this chapter:
  Turing machines represented as words
  $\rightsquigarrow$ Turing machines taking Turing machines as their input
- reductions: "embedding" a problem as a special case of another problem
- important method for proving undecidability:
  reduce from a known undecidable problem to a new problem