

# Theory of Computer Science

## C6. Context-free Languages: Closure & Decidability

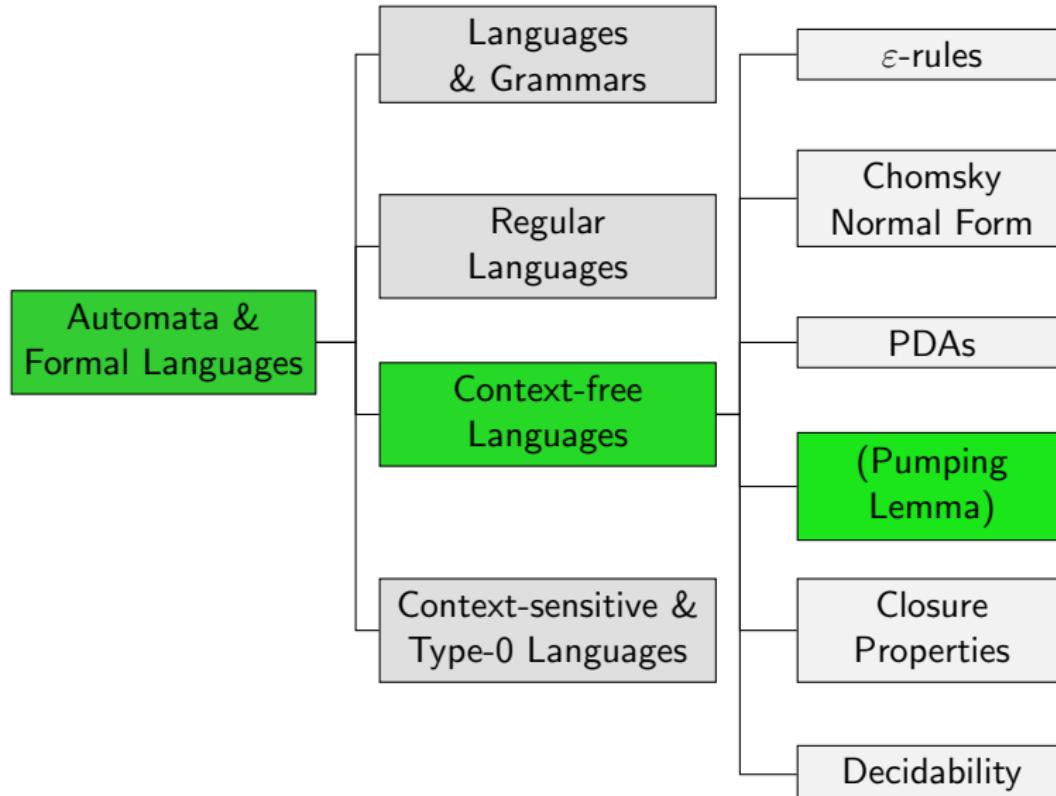
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# Pumping Lemma

# Overview



# Pumping Lemma for Context-free Languages



We used the pumping lemma from chapter C4 to show that a language is not regular. Is there a similar lemma for **context-free** languages?

# Pumping Lemma for Context-free Languages



We used the pumping lemma from chapter C4 to show that a language is not regular. Is there a similar lemma for **context-free** languages?

Yes!

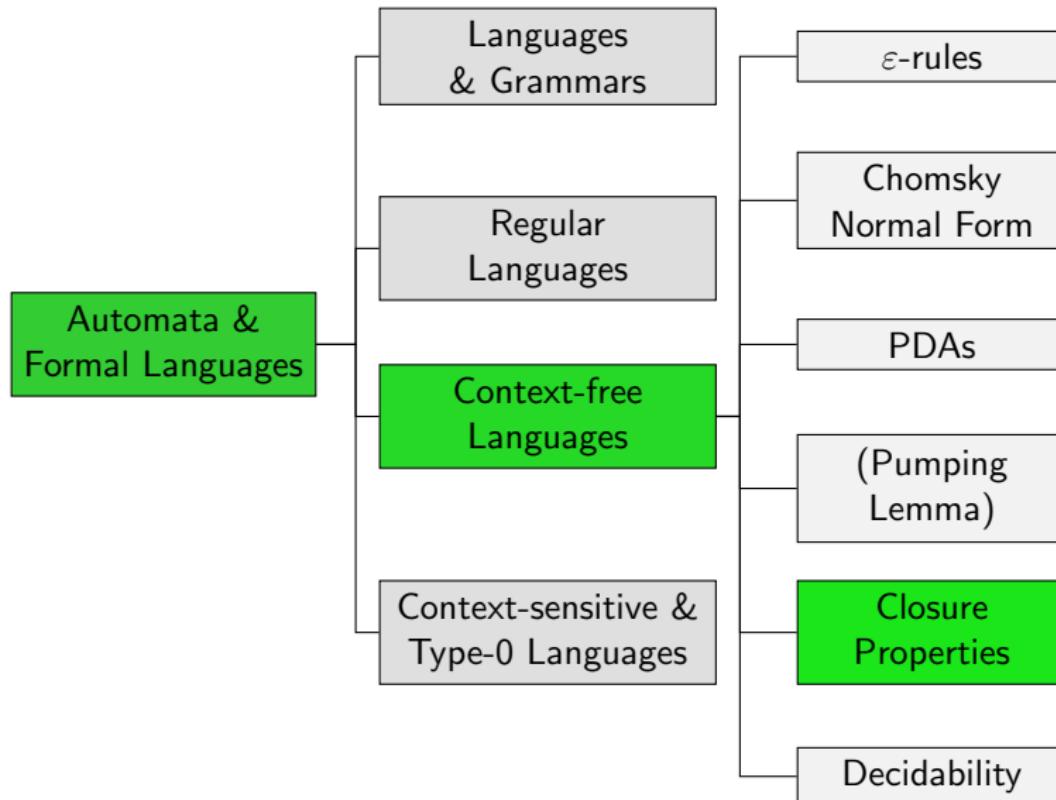
## Pumping Lemma for Context-free Languages

## Pumping lemma for context-free languages:

- It is possible to prove a variant of the **pumping lemma** for context-free languages.
- Pumping is more complex than for regular languages:
  - word is decomposed into the form  $uvwxy$  with  $|vx| \geq 1$ ,  $|vwx| \leq n$
  - pumped words have the form  $uv^iwx^i y$
- This allows us to prove that certain languages are **not context-free**.
- **example:**  $\{a^n b^n c^n \mid n \geq 1\}$  is not context-free (we will later use this without proof)

# Closure Properties

# Overview



# Closure under Union, Concatenation, Star

## Theorem

*The context-free languages are closed under:*

- *union*
- *concatenation*
- *star*

# Closure under Union, Concatenation, Star: Proof

## Proof.

### Closed under union:

Let  $G_1 = \langle \Sigma_1, V_1, P_1, S_1 \rangle$  and  $G_2 = \langle \Sigma_2, V_2, P_2, S_2 \rangle$  be context-free grammars. W.l.o.g.,  $V_1 \cap V_2 = \emptyset$ .

Then  $\langle \Sigma_1 \cup \Sigma_2, V_1 \cup V_2 \cup \{S\}, P_1 \cup P_2 \cup \{S \rightarrow S_1, S \rightarrow S_2\}, S \rangle$  (where  $S \notin V_1 \cup V_2$ ) is a context-free grammar for  $\mathcal{L}(G_1) \cup \mathcal{L}(G_2)$  (possibly requires rewriting  $\varepsilon$ -rules). ...

# Closure under Union, Concatenation, Star: Proof

Proof (continued).

Closed under concatenation:

Let  $G_1 = \langle \Sigma_1, V_1, P_1, S_1 \rangle$  and  $G_2 = \langle \Sigma_2, V_2, P_2, S_2 \rangle$  be context-free grammars. W.l.o.g.,  $V_1 \cap V_2 = \emptyset$ .

Then  $\langle \Sigma, V_1 \cup V_2 \cup \{S\}, P_1 \cup P_2 \cup \{S \rightarrow S_1 S_2\}, S \rangle$  (where  $S \notin V_1 \cup V_2$ ) is a context-free grammar for  $\mathcal{L}(G_1)\mathcal{L}(G_2)$  (possibly requires rewriting  $\varepsilon$ -rules).

...

# Closure under Union, Concatenation, Star: Proof

Proof (continued).

Closed under star:

Let  $G = \langle \Sigma, V, P, S \rangle$  be a context-free grammar  
where w.l.o.g.  $S$  never occurs on the right-hand side of a rule.

Then  $G' = \langle \Sigma, V \cup \{S'\}, P', S' \rangle$  with  $S' \notin V$  and  
 $P' = (P \cup \{S' \rightarrow \varepsilon, S' \rightarrow S, S' \rightarrow SS'\}) \setminus \{S \rightarrow \varepsilon\}$   
is a context-free grammar for  $\mathcal{L}(G)^*$  after rewriting  $\varepsilon$ -rules. □

# No Closure under Intersection or Complement

## Theorem

*The context-free languages are not closed under:*

- *intersection*
- *complement*

# No Closure under Intersection or Complement: Proof

## Proof.

Not closed under intersection:

The languages  $L_1 = \{a^i b^j c^j \mid i, j \geq 1\}$   
and  $L_2 = \{a^i b^j c^i \mid i, j \geq 1\}$  are context-free.

- For example,  $G_1 = \langle \{a, b, c\}, \{S, A, X\}, P, S \rangle$  with  
 $P = \{S \rightarrow AX, A \rightarrow a, A \rightarrow aA, X \rightarrow bc, X \rightarrow bXc\}$   
is a context-free grammar for  $L_1$ .
- For example,  $G_2 = \langle \{a, b, c\}, \{S, B\}, P, S \rangle$  with  
 $P = \{S \rightarrow aSc, S \rightarrow B, B \rightarrow b, B \rightarrow bB\}$   
is a context-free grammar for  $L_2$ .

Their intersection is  $L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 1\}$ .

We have remarked before that this language is not context-free.

...

# No Closure under Intersection or Complement: Proof

## Proof (continued).

Not closed under complement:

By contradiction: assume they were closed under complement.

Then they would also be closed under intersection  
because they are closed under union and

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}.$$

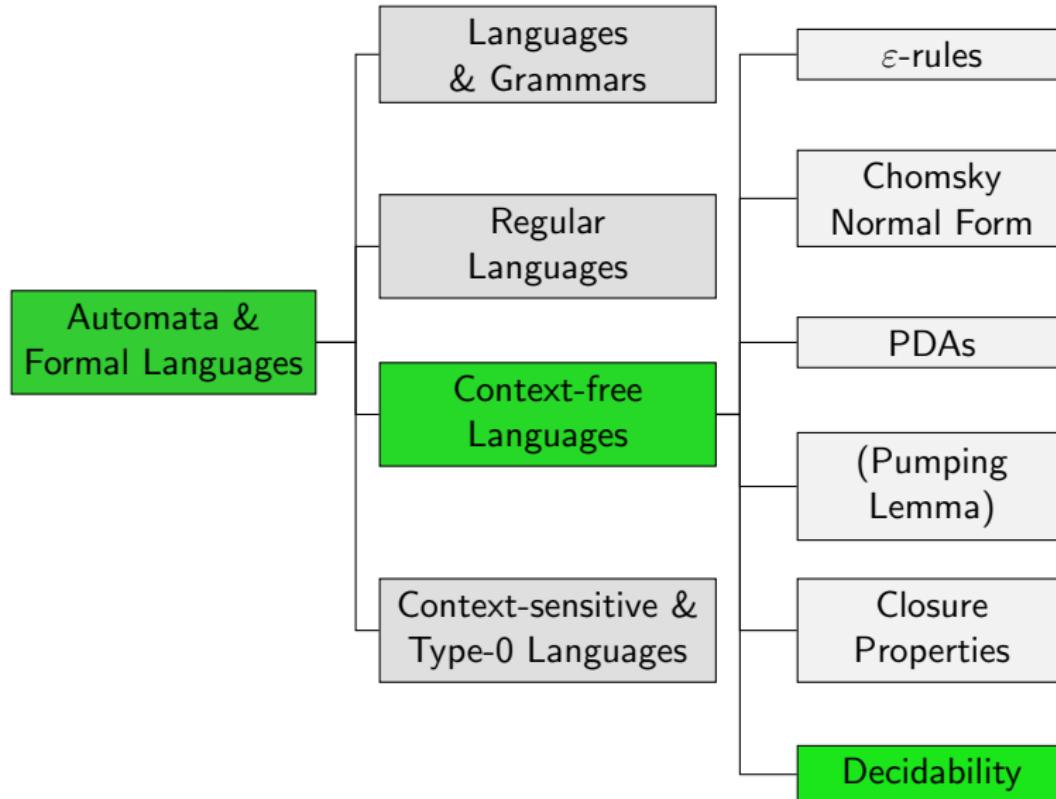
This is a contradiction because we showed  
that they are not closed under intersection. □

# Questions



# Decidability

# Overview



# Word Problem

## Definition (Word Problem for Context-free Languages)

The word problem  $P_{\in}$  for context-free languages is:

Given: context-free grammar  $G$  with alphabet  $\Sigma$   
and word  $w \in \Sigma^*$

Question: Is  $w \in \mathcal{L}(G)$ ?

# Decidability: Word Problem

## Theorem

*The word problem  $P_{\in}$  for context-free languages is **decidable**.*

## Proof.

If  $w = \varepsilon$ , then  $w \in \mathcal{L}(G)$  iff  $S \rightarrow \varepsilon$  with start variable  $S$  is a rule of  $G$ .

Since for all other rules  $w_l \rightarrow w_r$  of  $G$  we have  $|w_l| \leq |w_r|$ , the intermediate results when deriving a non-empty word never get shorter.

So it is possible to systematically consider all (finitely many) derivations of words up to length  $|w|$  and test whether they derive the word  $w$ .



**Note:** This is a terribly inefficient algorithm.

# Emptiness Problem

## Definition (Emptiness Problem for Context-free Languages)

The **emptiness problem**  $P_\emptyset$  for context-free languages is:

Given: context-free grammar  $G$

Question: Is  $\mathcal{L}(G) = \emptyset$ ?

# Decidability: Emptiness Problem

## Theorem

*The emptiness problem for context-free languages is **decidable**.*

## Proof.

Given a grammar  $G$ , determine all variables in  $G$  that allow deriving words that only consist of terminal symbols:

- First mark all variables  $A$  for which a rule  $A \rightarrow w$  exists such that  $w$  only consists of terminal symbols.
- Then mark all variables  $A$  for which a rule  $A \rightarrow w$  exists such that all nonterminal symbols in  $w$  are already marked.
- Repeat this process until no further markings are possible.

$\mathcal{L}(G)$  is empty iff the start variable is unmarked at the end of this process. □

# Finiteness Problem

## Definition (Finiteness Problem for Context-free Languages)

The **finiteness problem  $P_\infty$**  for context-free languages is:

Given: context-free grammar  $G$

Question: Is  $|\mathcal{L}(G)| < \infty$ ?

# Decidability: Finiteness Problem

## Theorem

*The finiteness problem for context-free languages is **decidable**.*

We omit the proof. A possible proof uses the pumping lemma for context-free languages.

### Proof sketch:

- We can compute certain bounds  $l, u \in \mathbb{N}_0$  for a given context-free grammar  $G$  such that  $\mathcal{L}(G)$  is infinite iff there exists  $w \in \mathcal{L}(G)$  with  $l \leq |w| \leq u$ .
- Hence we can decide finiteness by testing all (finitely many) such words by using an algorithm for the word problem.

# Intersection Problem

## Definition (Intersection Problem for Context-free Languages)

The **intersection problem  $P_{\cap}$**  for context-free languages is:

Given: context-free grammars  $G$  and  $G'$

Question: Is  $\mathcal{L}(G) \cap \mathcal{L}(G') = \emptyset$ ?

# Equivalence Problem

## Definition (Equivalence Problem for Context-free Languages)

The **equivalence problem  $P_=_$**  for context-free languages is:

Given: context-free grammars  $G$  and  $G'$

Question: Is  $\mathcal{L}(G) = \mathcal{L}(G')$ ?

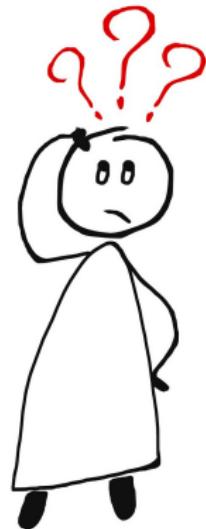
# Undecidability: Equivalence and Intersection Problem

## Theorem

*The equivalence problem for context-free languages  
and the intersection problem for context-free languages  
are **not decidable**.*

We cannot show this with the means currently available,  
but we will get back to this in Part D (computability theory).

# Questions



Questions?

# Summary

# Summary

- The context-free languages are **closed** under **union**, **concatenation** and **star**.
- The context-free languages are **not closed** under **intersection** or **complement**.
- The **word** problem, **emptiness** problem and **finiteness** problem for the class of context-free languages are **decidable**.
- The **equivalence** problem and **intersection** problem for the class of context-free languages are **not decidable**.

# Further Topics on Context-free Languages and PDAs

- With the **CYK-algorithm** (Cocke, Younger and Kasami) it is possible to decide  $w \in \mathcal{L}(G)$  in time  $O(|w|^3)$  for a grammar in Chomsky normal form and a word  $w$ .
- **Deterministic push-down automata** have the restriction  $|\delta(q, a, A)| + |\delta(q, \varepsilon, A)| \leq 1$  for all  $q \in Q, a \in \Sigma, A \in \Gamma$ . They accept with **end states** rather than empty stack.
- The languages accepted by deterministic PDAs are called **deterministic context-free languages**. They form a strict superset of the regular languages and a strict subset of the context-free languages.