Theory of Computer Science

C5. Context-free Languages: Normal Form and PDA

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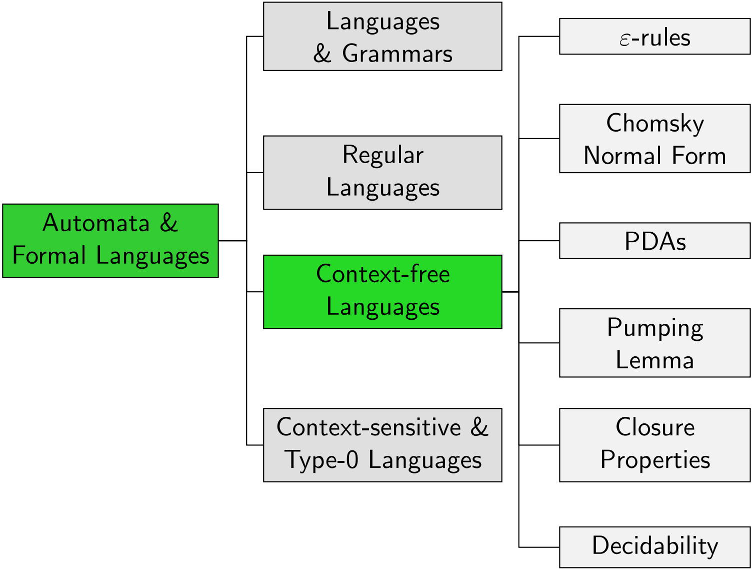
Overview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C5.1 Context-free Grammars and ε-Rules

Repetition: Context-free Grammars

Definition (Context-free Grammar)

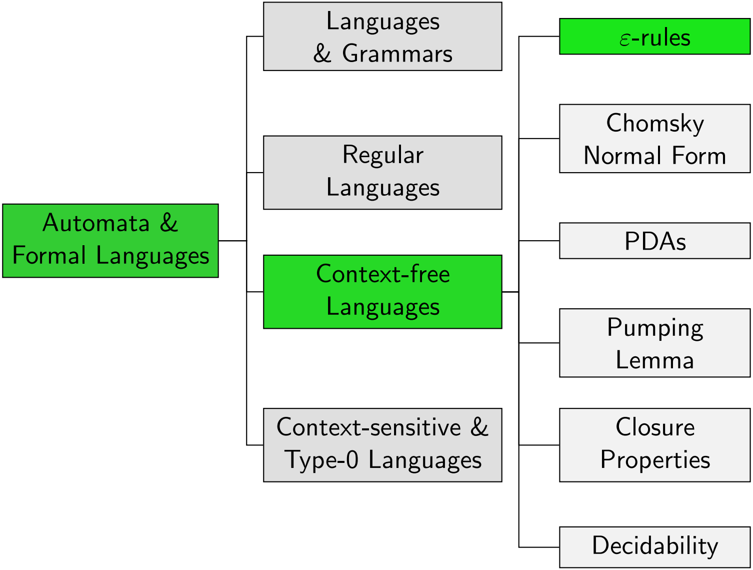
A context-free grammar is a 4-tuple $\langle \Sigma, V, P, S \rangle$ with

1. Σ finite alphabet of terminal symbols,
2. V finite set of variables (with $V \cap \Sigma = \emptyset$),
3. $P \subseteq (V \times (V \cup \Sigma)^+) \cup \{\langle S, \varepsilon \rangle\}$ finite set of rules,
4. If $S \to \varepsilon \in P$, then all other rules in $V \times ((V \setminus \{S\}) \cup \Sigma)^+$.
5. $S \in V$ start variable.

Rule $X \to \varepsilon$ is only allowed if $X = S$
and S never occurs on a right-hand side.

With regular grammars, this restriction could be lifted.
How about context-free grammars?

Overview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Reminder: Start Variable in Right-Hand Side of Rules

For every type-0 language L there is a grammar where the start variable does not occur on the right-hand side of any rule.

Theorem

For every grammar $G = \langle \Sigma, V, P, S \rangle$ there is a grammar $G' = \langle \Sigma, V', P', S \rangle$ with rules $P' \subseteq (V' \cup \Sigma)^+ \times (V' \setminus \{S\} \cup \Sigma)^*$ such that $\mathcal{L}(G) = \mathcal{L}(G')$.

In the proof we constructed a suitable grammar, where the rules in P' were not fundamentally different from the rules in P:

- for rules from $V \times (V \cup \Sigma)^+$, we only introduced additional rules from $V' \times (V' \cup \Sigma)^+$, and
- for rules from $V \times \varepsilon$, we only introduced rules from $V' \times \varepsilon$,

where $V' = V \cup \{S'\}$ for some new variable $S' \notin V$.

ε-Rules

Theorem

For every grammar G with rules $P \subseteq V \times (V \cup \Sigma)^*$ there is a context-free grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof.

Let $G = \langle \Sigma, V, P, S \rangle$ be a grammar with $P \subseteq V \times (V \cup \Sigma)^*$.

Let $G' = \langle \Sigma, V', P', S \rangle$ be a grammar with $\mathcal{L}(G) = \mathcal{L}(G')$ with $P' \subseteq V' \times ((V' \setminus S) \cup \Sigma)^*$.

Let $V_\varepsilon = \{A \in V' \mid A \Rightarrow^*_{G'} \varepsilon\}$. We can find this set V_ε by first collecting all variables A with rule $A \rightarrow \varepsilon \in P'$ and then successively adding additional variables B if there is a rule $B \rightarrow A_1 A_2 \dots A_k \in P'$ and the variables A_i are already in the set for all $1 \leq i \leq k$. ...

ε-Rules

Theorem

For every grammar G with rules $P \subseteq V \times (V \cup \Sigma)^*$ there is a context-free grammar G' with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

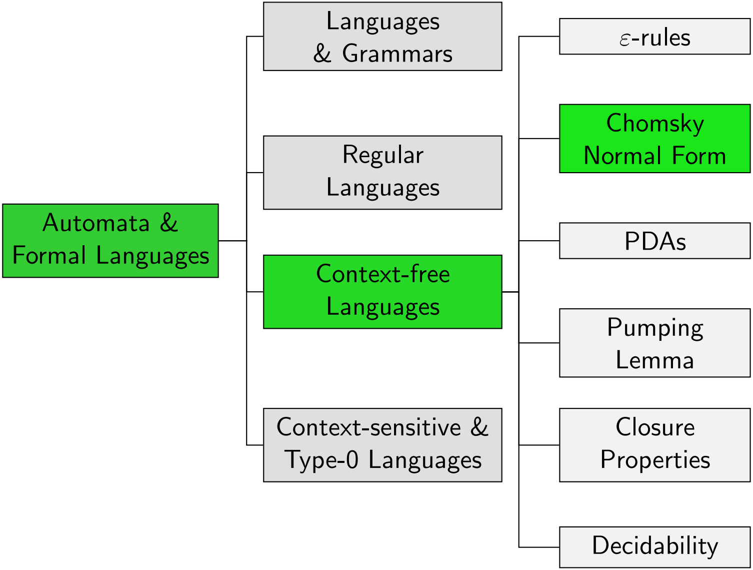
Let P'' be the rule set that is constructed from P' by

- adding rules that obviate the need for $A \to \varepsilon$ rules: for every existing rule $B \to w$ with $B \in V', w \in (V' \cup \Sigma)^+$, let I_ε be the set of positions where w contains a variable $A \in V_\varepsilon$. For every non-empty set $I' \subseteq I_\varepsilon$, add a new rule $B \to w'$, where w' is constructed from w by removing the variables at all positions in I'.
- removing all rules of the form $A \to \varepsilon$ ($A \neq S$).

Then $G'' = \langle \Sigma, V', P'', S \rangle$ is context-free and $\mathcal{L}(G) = \mathcal{L}(G'')$. □

C5.2 Chomsky Normal Form

Overview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Chomsky Normal Form: Motivation

As in logical formulas (and other kinds of structured objects), normal forms for grammars are useful:

- they show which aspects are critical for defining grammars and which ones are just syntactic sugar
- they allow proofs and algorithms to be restricted to a limited set of grammars (inputs): those in normal form

Hence we now consider a normal form for context-free grammars.

Chomsky Normal Form: Definition

Definition (Chomsky Normal Form)

A context-free grammar G is in Chomsky normal form (CNF) if all rules have one of the following three forms:

- $A \rightarrow BC$ with variables A, B, C, or
- $A \rightarrow a$ with variable A, terminal symbol a, or
- $S \rightarrow \varepsilon$ with start variable S.

German: Chomsky-Normalform

in short: rule set $P \subseteq (V \times (VV \cup \Sigma)) \cup \{\langle S, \varepsilon \rangle\}$

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof.

The following algorithm converts the rule set of G into CNF:

Step 1: Eliminate rules of the form $A \to B$ with variables A, B.

If there are sets of variables $\{B_1, \dots, B_k\}$ with rules $B_1 \to B_2, B_2 \to B_3, \dots, B_{k-1} \to B_k, B_k \to B_1$, then replace these variables by a new variable B.

Define a strict total order $<$ on the variables such that $A \to B \in P$ implies that $A < B$. Iterate from the largest to the smallest variable A and eliminate all rules of the form $A \to B$ while adding rules $A \to w$ for every rule $B \to w$ with $w \in (V \cup \Sigma)^+$. ...

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Step 2: Eliminate rules with terminal symbols on the right-hand side that do not have the form $A \rightarrow a$.

For every terminal symbol $a \in \Sigma$ add a new variable A_a and the rule $A_a \rightarrow a$.

Replace all terminal symbols in all rules that do not have the form $A \rightarrow a$ with the corresponding newly added variables. ...

Chomsky Normal Form: Theorem

Theorem

For every context-free grammar G there is a context-free grammar G' in Chomsky normal form with $\mathcal{L}(G) = \mathcal{L}(G')$.

Proof (continued).

Step 3: Eliminate rules of the form $A \rightarrow B_1 B_2 \dots B_k$ with $k > 2$

For every rule of the form $A \rightarrow B_1 B_2 \dots B_k$ with $k > 2$, add new variables $C_2, \dots, C_{k-1}$ and replace the rule with

$$\begin{aligned} A &\rightarrow B_1 C_2 \\ C_2 &\rightarrow B_2 C_3 \\ &\vdots \\ C_{k-1} &\rightarrow B_{k-1} B_k \end{aligned}$$

□

Chomsky Normal Form: Length of Derivations

Observation

Let G be a grammar in Chomsky normal form,
and let $w \in \mathcal{L}(G)$ be a non-empty word generated by G.

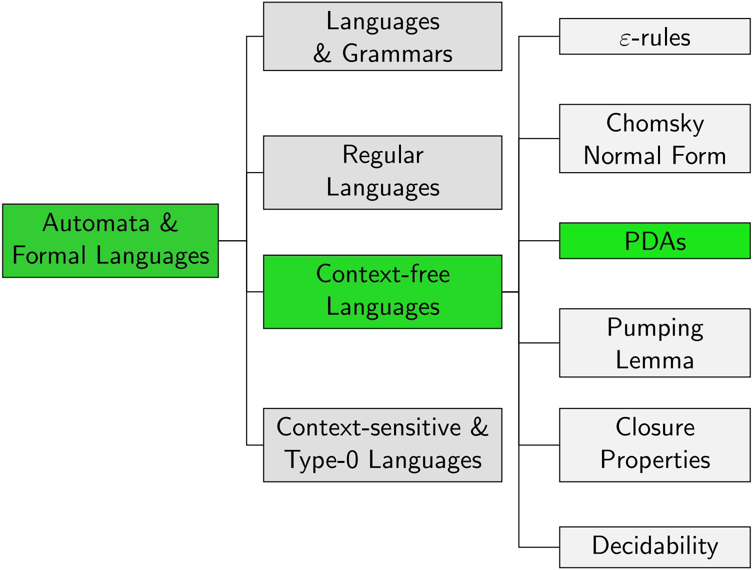
Then all derivations of w have exactly $2|w| - 1$ derivation steps.

Proof.

$\rightsquigarrow$ Exercises $\square$

C5.3 Push-Down Automata

Overview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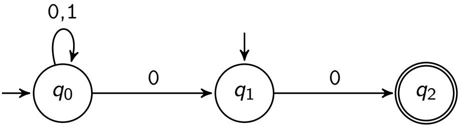
Limitations of Finite Automata

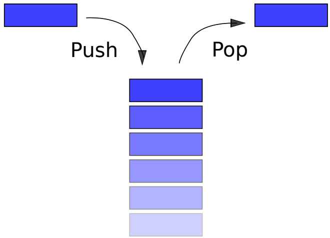


- Language L is regular.
  $\iff$ There is a finite automaton that accepts L.
- What information can a finite automaton "store" about the already read part of the word?
- Infinite memory would be required for
  $L = \{x_1 x_2 \ldots x_n x_n \ldots x_2 x_1 \mid n > 0, x_i \in \{\mathtt{a}, \mathtt{b}\}\}$.
- therefore: extension of the automata model with memory

Stack

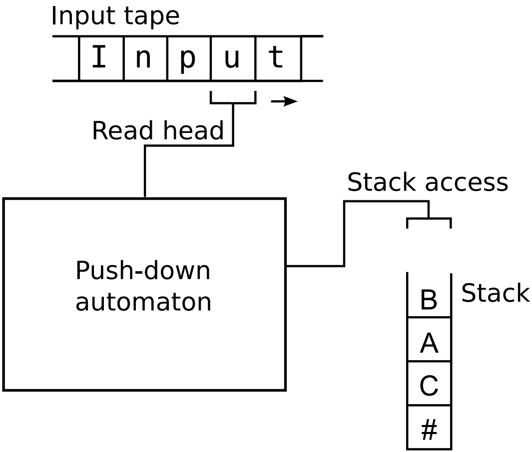
A stack is a data structure following the last-in-first-out (LIFO) principle supporting the following operations:

- push: puts an object on top of the stack
- pop: removes the object at the top of the stack
- peek: returns the top object without removing it



German: Keller, Stapel

Push-down Automata: Visually



German: Kellerautomat, Eingabeband, Lesekopf, Kellerzugriff

Push-down Automata: Definition

Definition (Push-down Automaton)

A push-down automaton (PDA) is a 6-tuple $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \# \rangle$ with

- Q finite set of states
- Σ the input alphabet
- Γ the stack alphabet
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to \mathcal{P}_f(Q \times \Gamma^*)$ the transition function (where $\mathcal{P}_f$ is the set of all finite subsets)
- $q_0 \in Q$ the start state
- $\# \in \Gamma$ the bottommost stack symbol

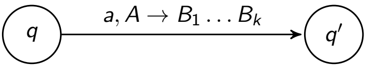
German: Kellerautomat, Eingabealphabet, Kelleralphabet, Überführungsfunktion

Push-down Automata: Transition Function

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \# \rangle$ be a push-down automaton.

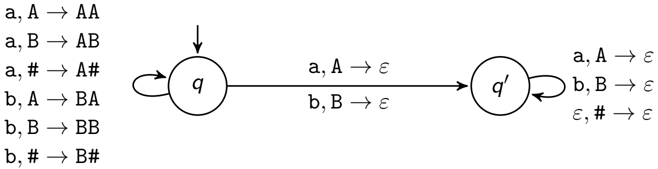
What is the Intuitive Meaning of the Transition Function δ?

- $\langle q', B_1 \dots B_k \rangle \in \delta(q, a, A)$: If M is in state q, reads symbol a and has A as the topmost stack symbol, then M can transition to q' in the next step while replacing A with $B_1 \dots B_k$ (afterwards B_1 is the topmost stack symbol)



- special case $a = \varepsilon$ is allowed (spontaneous transition)

Push-down Automata: Example



$M = \langle \{q, q'\}, \{\mathtt{a}, \mathtt{b}\}, \{\mathtt{A}, \mathtt{B}, \mathtt{\#}\}, \delta, q, \mathtt{\#} \rangle$ with

$\delta(q, \mathtt{a}, \mathtt{A}) = \{\langle q, \mathtt{AA}\rangle, \langle q', \varepsilon\rangle\}$ $\delta(q, \mathtt{b}, \mathtt{A}) = \{\langle q, \mathtt{BA}\rangle\}$ $\delta(q, \varepsilon, \mathtt{A}) = \emptyset$

$\delta(q, \mathtt{a}, \mathtt{B}) = \{\langle q, \mathtt{AB}\rangle\}$ $\delta(q, \mathtt{b}, \mathtt{B}) = \{\langle q, \mathtt{BB}\rangle, \langle q', \varepsilon\rangle\}$ $\delta(q, \varepsilon, \mathtt{B}) = \emptyset$

$\delta(q, \mathtt{a}, \mathtt{\#}) = \{\langle q, \mathtt{A\#}\rangle\}$ $\delta(q, \mathtt{b}, \mathtt{\#}) = \{\langle q, \mathtt{B\#}\rangle\}$ $\delta(q, \varepsilon, \mathtt{\#}) = \emptyset$

$\delta(q', \mathtt{a}, \mathtt{A}) = \{\langle q', \varepsilon\rangle\}$ $\delta(q', \mathtt{b}, \mathtt{A}) = \emptyset$ $\delta(q', \varepsilon, \mathtt{A}) = \emptyset$

$\delta(q', \mathtt{a}, \mathtt{B}) = \emptyset$ $\delta(q', \mathtt{b}, \mathtt{B}) = \{\langle q', \varepsilon\rangle\}$ $\delta(q', \varepsilon, \mathtt{B}) = \emptyset$

$\delta(q', \mathtt{a}, \mathtt{\#}) = \emptyset$ $\delta(q', \mathtt{b}, \mathtt{\#}) = \emptyset$ $\delta(q', \varepsilon, \mathtt{\#}) = \{\langle q', \varepsilon\rangle\}$

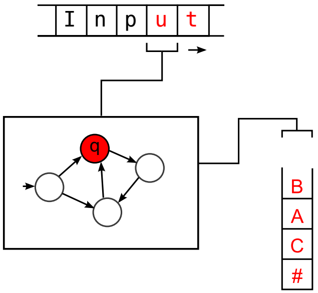
Push-down Automata: Configuration

Definition (Configuration of a Push-down Automaton)

A configuration of a push-down automaton $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \# \rangle$ is given by a triple $c \in Q \times \Sigma^* \times \Gamma^*$.

German: Konfiguration

Example



Configuration $\langle q, \mathtt{ut}, \mathtt{BAC\#} \rangle$.

Push-down Automata: Steps

Definition (Transition/Step of a Push-down Automaton)

We write $c \vdash_M c'$ if a push-down automaton $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \# \rangle$ can transition from configuration c to configuration c' in one step. Exactly the following transitions are possible:

$$\langle q, a_1 \dots a_n, A_1 \dots A_m \rangle \vdash_M \begin{cases} \langle q', a_2 \dots a_n, B_1 \dots B_k A_2 \dots A_m \rangle \\ \qquad \text{if } \langle q', B_1 \dots B_k \rangle \in \delta(q, a_1, A_1) \\ \langle q', a_1 a_2 \dots a_n, B_1 \dots B_k A_2 \dots A_m \rangle \\ \qquad \text{if } \langle q', B_1 \dots B_k \rangle \in \delta(q, \varepsilon, A_1) \end{cases}$$

German: Übergang

If M is clear from context, we only write $c \vdash c'$.

Push-down Automata: Reachability of Configurations

Definition (Reachable Configuration)

Configuration c' is reachable from configuration c in PDA M ($c \vdash_M^* c'$) if there are configurations $c_0, \dots, c_n$ ($n \geq 0$) where

- $c_0 = c$,
- $c_i \vdash_M c_{i+1}$ for all $i \in \{0, \dots, n-1\}$, and
- $c_n = c'$.

German: c' ist in M von c erreichbar

Push-down Automata: Recognized Words

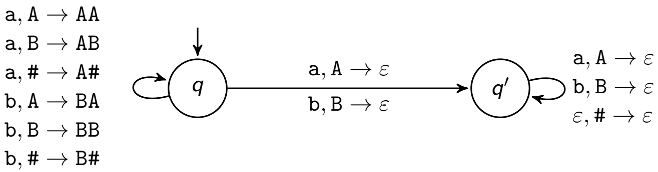
Definition (Recognized Word of a Push-down Automaton)

PDA $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \# \rangle$ recognizes the word $w = a_1 \dots a_n$ iff the configuration $\langle q, \varepsilon, \varepsilon \rangle$ (word processed and stack empty) for some $q \in Q$ is reachable from the start configuration $\langle q_0, w, \# \rangle$.

$$M \text{ recognizes } w \text{ iff } \langle q_0, w, \# \rangle \vdash_M^* \langle q, \varepsilon, \varepsilon \rangle \text{ for some } q \in Q.$$

German: M erkennt w, Startkonfiguration

Push-down Automata: Recognized Word Example



example: this PDA recognizes bbabbbabb ⇝ blackboard

Push-down Automata: Accepted Language

Definition (Accepted Language of a Push-down Automaton)

Let M be a push-down automaton with input alphabet Σ.
The language accepted by M is defined as

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ recognizes } w\}.$$

example: blackboard

PDAs Accept Exactly the Context-free Languages

Theorem

A language L is context-free if and only if L is accepted by a push-down automaton.

C5.4 Summary

Summary

- Every context-free language has a grammar in Chomsky normal form. All rules have form
  - $A \to BC$ with variables A, B, C, or
  - $A \to a$ with variable A, terminal symbol a, or
  - $S \to \varepsilon$ with start variable S.
- Push-down automata (PDAs) extend NFAs with memory.
- PDAs accept not with end states but with an empty stack.
- The languages accepted by PDAs are exactly the context-free languages.