Theory of Computer Science

A3. Proof Techniques

Gabriele Röger

University of Basel

February 17/19, 2020

Introduction

Mathematical Statements

Mathematical Statement

A mathematical statement consists of a set of preconditions and a set of conclusions.

The statement is true if the conclusions are true whenever the preconditions are true.

German: mathematische Aussage, Voraussetzung, Folgerung/Konklusion, wahr

Mathematical Statements

Mathematical Statement

A mathematical statement consists of a set of preconditions and a set of conclusions.

The statement is true if the conclusions are true whenever the preconditions are true.

German: mathematische Aussage, Voraussetzung, Folgerung/Konklusion, wahr

Notes:

- set of preconditions is sometimes empty
- often, “assumptions” is used instead of “preconditions”; slightly unfortunate because “assumption” is also used with another meaning ($\rightsquigarrow$ cf. indirect proofs)

Examples of Mathematical Statements

Examples (some true, some false):

- "Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd."
- "There exists an even prime number."
- "Let $p \in \mathbb{N}_0$ with $p \geq 3$ be a prime number. Then p is odd."
- "All prime numbers $p \geq 3$ are odd."
- "For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$"
- "The equation $a^k + b^k = c^k$ has infinitely many solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 2$."
- "The equation $a^k + b^k = c^k$ has no solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 3$."

What are the preconditions, what are the conclusions?

Proofs

Proof

A proof derives the correctness of a mathematical statement from a set of axioms and previously proven statements.

It consists of a sequence of proof steps, each of which directly follows from the axioms, previously proven statements and the preconditions of the statement,
ending with the conclusions of the theorem.

German: Beweis, Axiom, Beweisschritt

Disproofs

- A disproof (refutation) analogously shows that a given mathematical statement is false by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

German: Widerlegung

- Formally, disproofs are proofs of modified ("negated") statements.
- Be careful about how to negate a statement!

Proof Strategies

typical proof/disproof strategies:

1. "All $x \in S$ with the property P also have the property Q."
   "For all $x \in S$: if x has property P, then x has property Q."
   - To prove, assume you are given an arbitrary $x \in S$ that has the property P.
     Give a sequence of proof steps showing that x must have the property Q.
   - To disprove, find a counterexample, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

typical proof/disproof strategies:

2. "A is a subset of B."
   - To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
   - To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

typical proof/disproof strategies:

3. "For all $x \in S$: x has property P iff x has property Q."
   ("iff": "if and only if")
   - To prove, separately prove "if P then Q" and "if Q then P".
   - To disprove, disprove "if P then Q" or disprove "if Q then P".

   German: "iff" = gdw. ("genau dann, wenn")

Proof Strategies

typical proof/disproof strategies:

4. "$A = B$", where A and B are sets.
   - To prove, separately prove "$A \subseteq B$" and "$B \subseteq A$".
   - To disprove, disprove "$A \subseteq B$" or disprove "$B \subseteq A$".

Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contraposition
- mathematical induction
- structural induction

German: direkter Beweis, indirekter Beweis (Beweis durch Widerspruch), Kontraposition, vollständige Induktion, strukturelle Induktion

Direct Proof

Direct Proof

Direct Proof

Direct derivation of the statement by deducing or rewriting.

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ ($\subseteq$ part):

...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ ($\subseteq$ part):

Let $x \in A \cap (B \cup C)$. Then by the definition of $\cap$ it holds that $x \in A$ and $x \in B \cup C$.

...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ ($\subseteq$ part):

Let $x \in A \cap (B \cup C)$. Then by the definition of $\cap$ it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ ($\subseteq$ part):

Let $x \in A \cap (B \cup C)$. Then by the definition of $\cap$ it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies $x \in (A \cap B) \cup (A \cap C)$ ($\subseteq$ part):

Let $x \in A \cap (B \cup C)$. Then by the definition of $\cap$ it holds that $x \in A$ and $x \in B \cup C$.

We make a case distinction between $x \in B$ and $x \notin B$:

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

In both cases $x \in A \cap B$ or $x \in A \cap C$,
and we conclude $x \in (A \cap B) \cup (A \cap C)$. . . .

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

$\supseteq$ part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

$\supseteq$ part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.
The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

$\supseteq$ part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.
The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.
This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

. . .

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

$\supseteq$ part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

We make a case distinction between $x \in A \cap B$ and $x \notin A \cap B$:

If $x \in A \cap B$ then $x \in A$ and $x \in B$.
The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.
This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

In both cases we conclude $x \in A \cap (B \cup C)$. ...

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

We have shown that every element of $A \cap (B \cup C)$
is an element of $(A \cap B) \cup (A \cap C)$ and vice versa.
Thus, both sets are equal. □

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

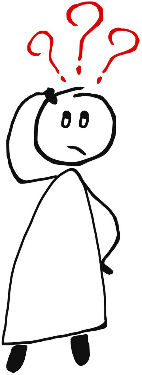
Proof.

Alternative:

$$\begin{aligned}
A \cap (B \cup C) &= \{x \mid x \in A \text{ and } x \in B \cup C\} \\
&= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\
&= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\
&= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\} \\
&= (A \cap B) \cup (A \cap C)
\end{aligned}$$

□

Questions



Questions?

Indirect Proof

Indirect Proof

Indirect Proof (Proof by Contradiction)

- Make an assumption that the statement is false.
- Derive a contradiction from the assumption together with the preconditions of the statement.
- This shows that the assumption must be false given the preconditions of the statement, and hence the original statement must be true.

German: Annahme, Widerspruch

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

□

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \ldots \cdot p_n + 1$.

□

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \ldots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \ldots \cdot p_n + 1$.

Since $m \geq 2$, it must have a prime factor.

Let p be such a prime factor.

□

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \ldots \cdot p_n + 1$.

Since $m \geq 2$, it must have a prime factor.
Let p be such a prime factor.

Since p is a prime number, p has to be in P.

□

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \ldots \cdot p_n + 1$.

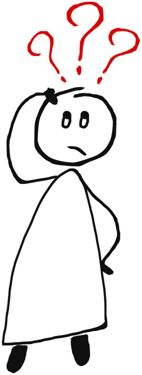
Since $m \geq 2$, it must have a prime factor.
Let p be such a prime factor.

Since p is a prime number, p has to be in P.

The number m is not divisible without remainder
by any of the numbers in P. Hence p is no factor of m.

$\rightsquigarrow$ Contradiction □

Questions



Questions?

Contraposition

Contraposition

(Proof by) Contraposition

Prove “If A, then B” by proving “If not B, then not A.”

German: (Beweis durch) Kontraposition

Contraposition

(Proof by) Contraposition

Prove “If A, then B” by proving “If not B, then not A.”

German: (Beweis durch) Kontraposition

Examples:

- Prove “For all $n \in \mathbb{N}_0$: if n^2 is odd, then n is odd” by proving “For all $n \in \mathbb{N}_0$, if n is even, then n^2 is even.”
- Prove “For all $n \in \mathbb{N}_0$: if n is not a square number, then $\sqrt{n}$ is irrational” by proving “For all $n \in \mathbb{N}_0$: if $\sqrt{n}$ is rational, then n is a square number.”

Mathematical Induction

Mathematical Induction

Mathematical Induction

Proof of a statement for all natural numbers n with $n \geq m$

- basis: proof of the statement for $n = m$
- induction hypothesis (IH):
  suppose that the statement is true for all k with $m \leq k \leq n$
- inductive step: proof of the statement for $n + 1$
  using the induction hypothesis

German: vollständige Induktion, Induktionsanfang, Induktionsvoraussetzung, Induktionsschritt

Mathematical Induction: Example I

Theorem

For all $n \in \mathbb{N}_0$ *with* $n \geq 1$*:* $\sum_{k=1}^{n}(2k-1) = n^2$

Mathematical Induction: Example I

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^{n}(2k-1) = n^2$

Proof.

Mathematical induction over n:

basis $n = 1$: $\sum_{k=1}^{1}(2k-1) = 2 - 1 = 1 = 1^2$

□

Mathematical Induction: Example I

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^{n}(2k-1) = n^2$

Proof.

Mathematical induction over n:

basis $n = 1$: $\sum_{k=1}^{1}(2k-1) = 2 - 1 = 1 = 1^2$

IH: $\sum_{k=1}^{m}(2k-1) = m^2$ for all $1 \leq m \leq n$

□

Mathematical Induction: Example I

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^{n}(2k-1) = n^2$

Proof.

Mathematical induction over n:

basis $n = 1$: $\sum_{k=1}^{1}(2k-1) = 2 - 1 = 1 = 1^2$

IH: $\sum_{k=1}^{m}(2k-1) = m^2$ for all $1 \leq m \leq n$

inductive step $n \rightarrow n+1$:

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1) &= \Big(\sum_{k=1}^{n}(2k-1)\Big) + 2(n+1) - 1 \\ &\overset{\text{IH}}{=} n^2 + 2(n+1) - 1 \\ &= n^2 + 2n + 1 = (n+1)^2\end{aligned}$$

□

Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers $p_1, \ldots, p_m$.

Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers $p_1, \ldots, p_m$.

Proof.

Mathematical Induction over n:

basis $n = 2$: trivially satisfied, since 2 is prime

...

Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers $p_1, \ldots, p_m$.

Proof.

Mathematical Induction over n:

basis $n = 2$: trivially satisfied, since 2 is prime

IH: Every natural number k with $2 \leq k \leq n$
can be written as a product of prime numbers. . . .

Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers $p_1, \ldots, p_m$.

Proof (continued).

inductive step $n \rightarrow n+1$:

- Case 1: $n+1$ is a prime number ⇝ trivial

□

Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$ with prime numbers $p_1, \ldots, p_m$.

Proof (continued).

inductive step $n \rightarrow n + 1$:

- Case 1: $n + 1$ is a prime number $\rightsquigarrow$ trivial
- Case 2: $n + 1$ is not a prime number.
  There are natural numbers $2 \leq q, r \leq n$ with $n + 1 = q \cdot r$.
  Using IH shows that there are prime numbers
  $q_1, \ldots, q_s$ with $q = q_1 \cdot \ldots \cdot q_s$ and
  $r_1, \ldots, r_t$ with $r = r_1 \cdot \ldots \cdot r_t$.
  Together this means $n + 1 = q_1 \cdot \ldots \cdot q_s \cdot r_1 \cdot \ldots \cdot r_t$.

□

Structural Induction

Inductively Defined Sets: Examples

Example (Natural Numbers)

The set $\mathbb{N}_0$ of natural numbers is inductively defined as follows:

- 0 is a natural number.
- If n is a natural number, then $n + 1$ is a natural number.

Inductively Defined Sets: Examples

Example (Natural Numbers)

The set $\mathbb{N}_0$ of natural numbers is inductively defined as follows:

- 0 is a natural number.
- If n is a natural number, then $n + 1$ is a natural number.

Example (Binary Tree)

The set $\mathcal{B}$ of binary trees is inductively defined as follows:

- $\square$ is a binary tree (a leaf)
- If L and R are binary trees, then $\langle L, \bigcirc, R \rangle$ is a binary tree (with inner node $\bigcirc$).

German: Binärbaum, Blatt, innerer Knoten

Inductively Defined Sets: Examples

Example (Natural Numbers)

The set $\mathbb{N}_0$ of natural numbers is inductively defined as follows:

- 0 is a natural number.
- If n is a natural number, then $n + 1$ is a natural number.

Example (Binary Tree)

The set $\mathcal{B}$ of binary trees is inductively defined as follows:

- $\square$ is a binary tree (a leaf)
- If L and R are binary trees, then $\langle L, \bigcirc, R\rangle$ is a binary tree (with inner node $\bigcirc$).

German: Binärbaum, Blatt, innerer Knoten

Implicit statement: all elements of the set can be constructed by finite application of these rules

Inductive Definition of a Set

Inductive Definition

A set M can be defined inductively by specifying

- basic elements that are contained in M
- construction rules of the form
  "Given some elements of M, another element of M can be constructed like this."

German: induktive Definition, Basiselemente, Konstruktionsregeln

Structural Induction

Structural Induction

Proof of statement for all elements of an inductively defined set

- basis: proof of the statement for the basic elements
- induction hypothesis (IH):
  suppose that the statement is true for some elements M
- inductive step: proof of the statement for elements constructed by applying a construction rule to M
  (one inductive step for each construction rule)

German: strukturelle Induktion, Induktionsanfang, Induktionsvoraussetzung, Induktionsschritt

Structural Induction: Example (1)

Definition (Leaves of a Binary Tree)

The number of leaves of a binary tree B, written $\mathit{leaves}(B)$, is defined as follows:

$$\begin{aligned} \mathit{leaves}(\square) &= 1 \\ \mathit{leaves}(\langle L, \bigcirc, R\rangle) &= \mathit{leaves}(L) + \mathit{leaves}(R) \end{aligned}$$

Definition (Inner Nodes of a Binary Tree)

The number of inner nodes of a binary tree B, written $\mathit{inner}(B)$, is defined as follows:

$$\begin{aligned} \mathit{inner}(\square) &= 0 \\ \mathit{inner}(\langle L, \bigcirc, R\rangle) &= \mathit{inner}(L) + \mathit{inner}(R) + 1 \end{aligned}$$

Structural Induction: Example (2)

Theorem

For all binary trees B: $\mathit{inner}(B) = \mathit{leaves}(B) - 1$.

Structural Induction: Example (2)

Theorem

For all binary trees B: $\mathit{inner}(B) = \mathit{leaves}(B) - 1$.

Proof.

induction basis:

$\mathit{inner}(\square) = 0 = 1 - 1 = \mathit{leaves}(\square) - 1$

$\rightsquigarrow$ statement is true for base case . . .

Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \bigcirc, R \rangle$, we may use that it is true for the subtrees L and R.

□

Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \bigcirc, R\rangle$, we may use that it is true for the subtrees L and R.

inductive step for $B = \langle L, \bigcirc, R\rangle$:

$$\begin{aligned} \mathit{inner}(B) &= \mathit{inner}(L) + \mathit{inner}(R) + 1 \\ &\overset{\text{IH}}{=} (\mathit{leaves}(L) - 1) + (\mathit{leaves}(R) - 1) + 1 \\ &= \mathit{leaves}(L) + \mathit{leaves}(R) - 1 = \mathit{leaves}(B) - 1 \end{aligned}$$

□

Structural Induction: Exercise

Definition (Height of a Binary Tree)

The height of a binary tree B, written $height(B)$, is defined as follows:

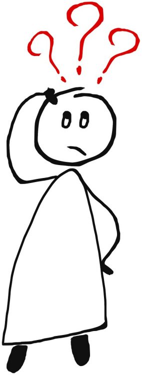
$$height(\square) = 0$$
$$height(\langle L, \bigcirc, R\rangle) = \max\{height(L), height(R)\} + 1$$

Prove by structural induction:

Theorem

For all binary trees B: $leaves(B) \leq 2^{height(B)}$.

Questions



Questions?

Summary

Summary

- A proof is based on axioms and previously proven statements.
- Individual proof steps must be obvious derivations.
- direct proof: sequence of derivations or rewriting
- indirect proof: refute the negated statement
- contraposition: prove “$A \Rightarrow B$” as “not $B \Rightarrow$ not A”
- mathematical induction: prove statement for a starting point and show that it always carries over to the next number
- structural induction: generalization of mathematical induction to arbitrary recursive structures