

Theory of Computer Science

A2. Mathematical Foundations

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February 17, 2020

Sets, Tuples, Relations

Sets

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each object contained **at most once**

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 - **explicit**, listing all elements, e. g. $A = \{1, 2, 3\}$
 - **implicit**, specifying a **property** characterizing all elements,
e. g. $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
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e. g. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

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German: Menge, Element, leere Menge, Mächtigkeit/Kardinalität

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i. e., every element of A is an element of B
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- **power set $\mathcal{P}(M)$** : set of all subsets of M
e. g., $\mathcal{P}(\{a, b\}) =$

German: Teilmenge, echte Teilmenge, Potenzmenge

Set Operations

- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

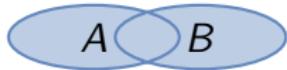


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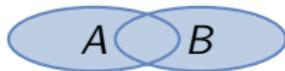


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- complement $\bar{A} = B \setminus A$, where $A \subseteq B$ and B is the set of all considered objects (in a given context)

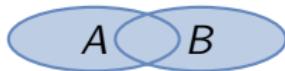


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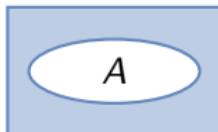
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German: Schnitt, Vereinigung, Differenz, Komplement

Tuples

- ***k*-tuple**: ordered sequence of k objects
- written (o_1, \dots, o_k) or $\langle o_1, \dots, o_k \rangle$
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- objects contained in tuples are called **components**
- terminology:
 - $k = 2$: (ordered) pair
 - $k = 3$: triple
 - more rarely: quadruple, quintuple, sextuple, septuple, ...
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German: *k*-Tupel, Komponente, Paar, Tripel

Cartesian Product

- for sets M_1, M_2, \dots, M_n , the **Cartesian product**
 $M_1 \times \dots \times M_n$ is the set
$$M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}.$$
- Example: $M_1 = \{a, b, c\}, M_2 = \{1, 2\},$
$$M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$$

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- special case: $M^k = M \times \dots \times M$ (k times)
- example with $M = \{1, 2\}$:
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German: kartesisches Produkt

Relations

- an n -ary **relation** R over the sets M_1, \dots, M_n is a subset of their Cartesian product: $R \subseteq M_1 \times \dots \times M_n$.
- example with $M = \{1, 2\}$:
 $R_{\leq} \subseteq M^2$ as $R_{\leq} = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle\}$

German: (n -stellige) Relation

Exercise

Consider $S = \mathcal{P}(\{1, 2\}) \times \{a, b\}$.

- ① Write down three different elements of S .
- ② What is $|S|$?

Functions

Functions

Definition (Total Function)

A (total) **function** $f : D \rightarrow C$ (with sets D, C)
maps **every value** of its **domain** D
to **exactly one value** of its **codomain** C .

German: (totale) Funktion, Definitionsbereich, Wertebereich

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- $\text{add} : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$ with $\text{add}(x, y) = x + y$
- $\text{add}_{\mathbb{R}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $\text{add}_{\mathbb{R}}(x, y) = x + y$

Functions: Example

Example

Let $Q = \{q_0, q_1, q_2, q_e\}$ and $\Gamma = \{0, 1, \square\}$.

Define $\delta : (Q \setminus \{q_e\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, N\}$ by

δ	0	1	\square
q_0	$\langle q_0, 0, R \rangle$	$\langle q_0, 1, R \rangle$	$\langle q_1, \square, L \rangle$
q_1	$\langle q_2, 1, L \rangle$	$\langle q_1, 0, L \rangle$	$\langle q_e, 1, N \rangle$
q_2	$\langle q_2, 0, L \rangle$	$\langle q_2, 1, L \rangle$	$\langle q_e, \square, R \rangle$

Then, e. g., $\delta(q_0, 1) = \langle q_0, 1, R \rangle$

Partial Functions

Definition (Partial Function)

A **partial function** $f : X \rightarrow_p Y$ maps every value in X to **at most** one value in Y .

If f does not map $x \in X$ to any value in Y , then f is **undefined** for x .

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Example

$f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow_p \mathbb{N}_0$ with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

Exercise

Let $V = \{X, Y, Z\}$, $\Sigma = \{a, b, c\}$ and $Q = \{q_1, q_2\}$ be three sets.

- 1 Specify a non-trivial example for a partial function

$$\delta : Q \times \Sigma \rightarrow_p \mathcal{P}(Q).$$

- 2 Specify a non-trivial example for a relation

$$P \subseteq (V \cup \Sigma)^2 \times V^2.$$

Summary

Summary

- **sets**: unordered, contain every element at most once
- **tuples**: ordered, can contain the same object multiple times
- **Cartesian product**: $M_1 \times \cdots \times M_n$ set of all n -tuples where the i -th component is in M_i ;
- **function** $f : X \rightarrow Y$ maps every value in X to exactly one value in Y
- **partial function** $g : X \rightarrow_p Y$ may be undefined for some values in X