

# Theory of Computer Science

## A2. Mathematical Foundations

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## A2.1 Sets, Tuples, Relations

## A2.2 Functions

## A2.3 Summary

# A2.1 Sets, Tuples, Relations

# Sets

- ▶ **set**: **unordered collection** of distinguishable objects; each object contained **at most once**
- ▶ notations:
  - ▶ **explicit**, listing all elements, e. g.  $A = \{1, 2, 3\}$
  - ▶ **implicit**, specifying a **property** characterizing all elements, e. g.  $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
  - ▶ **implicit**, as a **sequence with dots**, e. g.  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- ▶  $e \in M$ :  $e$  is in set  $M$  (an **element** of the set)
- ▶  $e \notin M$ :  $e$  is not in set  $M$
- ▶ **empty set**  $\emptyset = \{\}$
- ▶ **cardinality**  $|M|$  of a finite set  $M$ : number of elements in  $M$

**German**: Menge, Element, leere Menge, Mächtigkeit/Kardinalität

# Sets

- ▶  $A \subseteq B$ :  $A$  is a **subset** of  $B$ ,  
i. e., every element of  $A$  is an element of  $B$
- ▶  $A \subset B$ :  $A$  is a **strict subset** of  $B$ ,  
i. e.,  $A \subseteq B$  and  $A \neq B$ .
- ▶ **power set**  $\mathcal{P}(M)$ : set of all subsets of  $M$   
e. g.,  $\mathcal{P}(\{a, b\}) =$

**German:** Teilmenge, echte Teilmenge, Potenzmenge

# Set Operations

- ▶ **intersection**  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



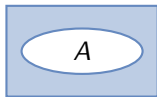
- ▶ **union**  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



- ▶ **difference**  $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



- ▶ **complement**  $\bar{A} = B \setminus A$ , where  $A \subseteq B$  and  $B$  is the set of all considered objects (in a given context)



**German:** Schnitt, Vereinigung, Differenz, Komplement

# Tuples

- ▶  **$k$ -tuple**: ordered sequence of  $k$  objects
- ▶ written  $(o_1, \dots, o_k)$  or  $\langle o_1, \dots, o_k \rangle$
- ▶ unlike sets, **order matters** ( $\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$ )
- ▶ objects may occur multiple times in a tuple
- ▶ objects contained in tuples are called **components**
- ▶ terminology:
  - ▶  $k = 2$ : (ordered) pair
  - ▶  $k = 3$ : triple
  - ▶ more rarely: quadruple, quintuple, sextuple, septuple, ...
- ▶ if  $k$  is clear from context (or does not matter), often just called **tuple**

**German:**  $k$ -Tupel, Komponente, Paar, Tripel

# Cartesian Product

- ▶ for sets  $M_1, M_2, \dots, M_n$ , the **Cartesian product**  $M_1 \times \dots \times M_n$  is the set  $M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}$ .
- ▶ Example:  $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$ ,  
 $M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$
- ▶ special case:  $M^k = M \times \dots \times M$  ( $k$  times)
- ▶ example with  $M = \{1, 2\}$ :  
 $M^2 = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$

German: kartesisches Produkt



# Relations

- ▶ an  $n$ -ary **relation**  $R$  over the sets  $M_1, \dots, M_n$  is a subset of their Cartesian product:  $R \subseteq M_1 \times \dots \times M_n$ .
- ▶ example with  $M = \{1, 2\}$ :  
 $R_{\leq} \subseteq M^2$  as  $R_{\leq} = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle\}$

**German:** ( $n$ -stellige) Relation

# Exercise

Consider  $S = \mathcal{P}(\{1, 2\}) \times \{a, b\}$ .

- 1 Write down three different elements of  $S$ .
- 2 What is  $|S|$ ?

## A2.2 Functions

# Functions

## Definition (Total Function)

A (total) **function**  $f : D \rightarrow C$  (with sets  $D, C$ ) maps **every value** of its **domain**  $D$  to **exactly one value** of its **codomain**  $C$ .

**German:** (totale) Funktion, Definitionsbereich, Wertebereich

## Example

- ▶  $square : \mathbb{Z} \rightarrow \mathbb{Z}$  with  $square(x) = x^2$
- ▶  $add : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0$  with  $add(x, y) = x + y$
- ▶  $add_{\mathbb{R}} : \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $add_{\mathbb{R}}(x, y) = x + y$

# Functions: Example

## Example

Let  $Q = \{q_0, q_1, q_2, q_e\}$  and  $\Gamma = \{0, 1, \square\}$ .

Define  $\delta : (Q \setminus \{q_e\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, N\}$  by

$\delta$	0	1	$\square$
$q_0$	$\langle q_0, 0, R \rangle$	$\langle q_0, 1, R \rangle$	$\langle q_1, \square, L \rangle$
$q_1$	$\langle q_2, 1, L \rangle$	$\langle q_1, 0, L \rangle$	$\langle q_e, 1, N \rangle$
$q_2$	$\langle q_2, 0, L \rangle$	$\langle q_2, 1, L \rangle$	$\langle q_e, \square, R \rangle$

Then, e. g.,  $\delta(q_0, 1) = \langle q_0, 1, R \rangle$

# Partial Functions

## Definition (Partial Function)

A **partial function**  $f : X \rightarrow_p Y$  maps every value in  $X$  to **at most** one value in  $Y$ .

If  $f$  does not map  $x \in X$  to any value in  $Y$ , then  $f$  is **undefined** for  $x$ .

**German:** partielle Funktion

## Example

$f : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow_p \mathbb{N}_0$  with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$

## Exercise

Let  $V = \{X, Y, Z\}$ ,  $\Sigma = \{a, b, c\}$  and  $Q = \{q_1, q_2\}$  be three sets.

- 1 Specify a non-trivial example for a partial function

$$\delta : Q \times \Sigma \rightarrow_p \mathcal{P}(Q).$$

- 2 Specify a non-trivial example for a relation

$$P \subseteq (V \cup \Sigma)^2 \times V^2.$$

## A2.3 Summary



# Summary

- ▶ **sets:** unordered, contain every element at most once
- ▶ **tuples:** ordered, can contain the same object multiple times
- ▶ **Cartesian product:**  $M_1 \times \cdots \times M_n$  set of all  $n$ -tuples where the  $i$ -th component is in  $M_i$
- ▶ **function**  $f : X \rightarrow Y$  maps every value in  $X$  to exactly one value in  $Y$
- ▶ **partial function**  $g : X \rightarrow_p Y$  may be undefined for some values in  $X$