Theory of Computer Science

G. Röger
Spring Term 2020

University of Basel
Computer Science

Exercise Sheet 4 — Solutions

Exercise 4.1 (Predicate logic, 2 + 2 marks)

(a) Show for arbitrary predicate logic formulas φ and ψ that

$$(\forall x\varphi \lor \forall x\psi) \models \forall x(\varphi \lor \psi).$$

Solution:

Let $\mathcal{I}$ be an interpretation and α be a variable assignment such that $\mathcal{I}, \alpha \models (\forall x\varphi \lor \forall x\psi)$. By the semantics of the disjunction, it is the case that $\mathcal{I}, \alpha \models \forall x\varphi$ or $\mathcal{I}, \alpha \models \forall x\psi$.

Case 1: $\mathcal{I}, \alpha \models \forall x\varphi$

In this case it is true for all objects u in the universe that $\mathcal{I}, \alpha[x := u] \models \varphi$. But then $\mathcal{I}, \alpha[x := u] \models (\varphi \lor \chi)$ for arbitrary formulas χ by the semantics of the disjunction. With $\chi := \psi$, we get $\mathcal{I}, \alpha[x := u] \models (\varphi \lor \psi)$ for all objects u of the universe. We can conclude with the semantics of $\forall$ that $\mathcal{I}, \alpha \models \forall x(\varphi \lor \psi)$.

Case 2: $\mathcal{I}, \alpha \models \forall x\varphi$

We can argue analogously to case 1 that $\mathcal{I}, \alpha \models \forall x(\varphi \lor \psi)$.

Hence, every model of $(\forall x\varphi \lor \forall x\psi)$ is a model of $\forall x(\varphi \lor \psi)$ and we conclude that $(\forall x\varphi \lor \forall x\psi) \models \forall x(\varphi \lor \psi)$.

(b) Show that in general it is *not* the case that

$$(\forall x\varphi \lor \forall x\psi) \equiv \forall x(\varphi \lor \psi)$$

Specify a counter example with the following signature: $\mathcal{S} = \langle \{x\}, \{\}, \{\}, \{P, Q\} \rangle$, where $ar(P) = ar(Q) = 1$.

Solution:

Consider interpretation $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with $U = \{u_1, u_2\}$, $P^{\mathcal{I}} = \{u_1\}$ and $Q^{\mathcal{I}} = \{u_2\}$. For arbitrary variable assignments α, it holds that $\mathcal{I}, \alpha \models \forall x(P(x) \lor Q(x))$ because $\mathcal{I}, \alpha[x := u_1] \models P(x)$ and $\mathcal{I}, \alpha[x := u_2] \models Q(x)$. However, $\mathcal{I}, \alpha \not\models \forall x P(x)$ because $\mathcal{I}, \alpha[x := u_2] \not\models P(x)$ and – analogously – $\mathcal{I}, \alpha \not\models \forall x Q(x)$ because $\mathcal{I}, \alpha[x := u_1] \not\models Q(x)$. Hence we can conclude that $\mathcal{I}, \alpha \not\models (\forall x P(x) \lor \forall x Q(x))$ and have seen an example, where a model of $\forall x(P(x) \lor Q(x))$ is *not* a model of $(\forall x P(x) \lor \forall x Q(x))$.

With $\varphi = P(x)$ and $\psi = Q(x)$, we see that in general it is not the case that $\forall x(\varphi \lor \psi) \models (\forall x\varphi \lor \forall x\psi)$ and hence also the logical equivalence cannot hold in general.

Exercise 4.2 (Predicate logic, 1 mark)

Use equivalence transformations to bring the following formula in negation normal form. For this purpose, move negation symbols inwards by using DeMorgan's rule or equivalence $\neg\forall x\varphi \equiv \exists x\neg\varphi$ and $\neg\exists x\varphi \equiv \forall x\neg\varphi$, or eliminate them with double negation.

$$\varphi = \neg\forall x((P(x) \lor \neg Q(x, \mathrm{c})) \land \exists y(P(x) \to Q(y, x)))$$

Solution:

$$
\begin{aligned}
\varphi &= \neg\forall x((P(x) \lor \neg Q(x,\mathrm{c})) \land \exists y(P(x) \to Q(y,x)))\\
&\equiv \exists x\neg((P(x) \lor \neg Q(x,\mathrm{c})) \land \exists y(P(x) \to Q(y,x)))\\
&\equiv \exists x(\neg(P(x) \lor \neg Q(x,\mathrm{c})) \lor \neg\exists y(P(x) \to Q(y,x)))\\
&\equiv \exists x((\neg P(x) \land \neg\neg Q(x,\mathrm{c})) \lor \neg\exists y(P(x) \to Q(y,x)))\\
&\equiv \exists x((\neg P(x) \land Q(x,\mathrm{c})) \lor \neg\exists y(P(x) \to Q(y,x)))\\
&\equiv \exists x((\neg P(x) \land Q(x,\mathrm{c})) \lor \forall y\neg(P(x) \to Q(y,x)))\\
&\equiv \exists x((\neg P(x) \land Q(x,\mathrm{c})) \lor \forall y\neg(\neg P(x) \lor Q(y,x)))\\
&\equiv \exists x((\neg P(x) \land Q(x,\mathrm{c})) \lor \forall y(\neg\neg P(x) \land \neg Q(y,x)))\\
&\equiv \exists x((\neg P(x) \land Q(x,\mathrm{c})) \lor \forall y(P(x) \land \neg Q(y,x)))
\end{aligned}
$$

Exercise 4.3 (Formal languages and grammars, 1+3+1 marks)
Consider the following formal language over $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$:

$$L = \{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^{2n} \mid n \geq 0, m \geq 0\}$$

(a) Is ε an element of L? Justify your answer.

Solution:

Yes, L contains the word $\mathtt{a}^n\mathtt{b}^m\mathtt{c}^{2n}$ for any $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, in particular for $n = 0$ and $m = 0$ the word $\mathtt{a}^0\mathtt{b}^0\mathtt{c}^{2\cdot 0} = \mathtt{a}^0\mathtt{b}^0\mathtt{c}^0 = \varepsilon$.

(b) Specify a *complete description* of a formal grammar G that generates L (i.e., $\mathcal{L}(G) = L$). A formal grammar is a four tuple $G = \langle \Sigma, V, P, S \rangle$, remember to define all components of this tuple.

Solution:

$G = (\Sigma, V, P, S)$ with $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$, $V = \{\mathrm{S}, \mathrm{A}, \mathrm{B}, \mathrm{C}\}$ and the following rules in the set P:

$\mathrm{S} \to \varepsilon$ $\quad$ $\mathrm{S} \to \mathrm{ACC}$ $\quad$ $\mathrm{S} \to \mathrm{B}$ $\quad$ $\mathrm{ACC} \to \mathrm{AACCCC}$ $\quad$ $\mathrm{ACC} \to \mathrm{ABCC}$ $\quad$ $\mathrm{B} \to \mathrm{BB}$
$\mathrm{A} \to \mathtt{a}$ $\quad$ $\mathrm{B} \to \mathtt{b}$ $\quad$ $\mathrm{C} \to \mathtt{c}$

(c) Which types (in the Chomsky-Hierarchy) is your formal grammar part of? You don't have to prove your answers.

Solution:

The above specified formal grammar is part of the following types in the Chomsky-Hierarchy:

- Type 0, since *all* formal grammars are of Type 0.
- Type 1, since for all rules in P (except for $\mathrm{S} \to \varepsilon$) the left side is shorter or equally long as the right side. Although the right side of $\mathrm{S} \to \varepsilon$ is shorter than the left side (since ε has length 0), G is still of Type 1 since S is the start symbol and never occurs in the right side of any rule.

The grammar is not of Type 2, since for example the left side of the rule $\mathrm{ACC} \to \mathrm{ABCC}$ does not consist of a single variable. Since it is not of Type 2, it cannot be of Type 3 either.