Theory of Computer Science

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Exercise Sheet 3 — Solutions

Exercise 3.1 (Equivalences; 1.5+1.5 marks)

To formally prove the correctness of a calculus one needs to show for every rule

$$\frac{\varphi_1, \dots, \varphi_n}{\psi}$$

that $\{\varphi_1, \dots, \varphi_n\} \models \psi$.

(a) Prove the correctness of modus tollens.

Solution:

Let $\mathcal{I}$ be an arbitrary model of formulas $\neg\psi$ and $(\varphi \rightarrow \psi)$. From $\mathcal{I} \models (\varphi \rightarrow \psi)$ we know that $\mathcal{I} \models \psi$ if $\mathcal{I} \models \varphi$. Since $\mathcal{I} \models \neg\psi$ holds, we know that $\mathcal{I} \not\models \psi$ and we conclude that $\mathcal{I} \not\models \varphi$. This implies that $\mathcal{I} \models \neg\varphi$. Overall, we have shown that every model of $\neg\psi$ and $(\varphi \rightarrow \psi)$ is also a model of $\neg\varphi$ and, thus, $\{\neg\psi, (\varphi \rightarrow \psi)\} \models \neg\varphi$.

(b) Prove the correctness of a new inference rule

$$\frac{(\varphi \lor \chi), (\psi \lor \neg\chi)}{(\varphi \lor \psi)}.$$

Solution:

Let $\mathcal{I}$ be an arbitrary model of formulas $(\varphi \lor \chi)$ and $(\psi \lor \neg\chi)$. We distinguish two cases:

Case 1 ($\mathcal{I} \models \chi$): From $\mathcal{I} \models (\psi \lor \neg\chi)$ it follows that $\mathcal{I} \models \psi$ oder $\mathcal{I} \models \neg\chi$ holds. But due to $\mathcal{I} \models \chi$ it cannot be the case that $\mathcal{I} \models \neg\chi$ holds, so $\mathcal{I} \models \psi$ must be true. This implies that $\mathcal{I} \models (\varphi \lor \psi)$.

Case 2 ($\mathcal{I} \not\models \chi$): From $\mathcal{I} \models (\varphi \lor \chi)$ follows with the same argumentation as above that $\mathcal{I} \models \varphi$ holds. This implies that $\mathcal{I} \models (\varphi \lor \psi)$ is true.

We have shown that every model of formulas $(\varphi \lor \chi)$ and $(\psi \lor \neg\chi)$ is a model of $(\varphi \lor \psi)$. Hence it holds that $\{(\varphi \lor \chi), (\psi \lor \neg\chi)\} \models (\varphi \lor \psi)$.

Exercise 3.2 (Resolution Calculus; 2 Points)

Consider the following knowledge base

$$\mathrm{KB} = \{(A \leftrightarrow \neg D), (\neg A \rightarrow (B \lor C)), ((A \rightarrow E) \land (B \lor C \lor F)), (E \rightarrow (F \rightarrow (B \lor C))), (C \rightarrow G), (G \rightarrow \neg C)\}.$$

Use the resolution calculus to show that $\mathrm{KB} \models (B \land \neg C)$.

Solution:

In order to show that $KB \models (B \land \neg C)$, we prove that $KB' = KB \cup \{\neg(B \land \neg C)\}$ is unsatisfiable. Since we want to apply resolution we first need to transform KB' into conjunctive normal form:

Formulas (and Equivalences)	Clauses
$(A \leftrightarrow \neg D)$ $\equiv ((A \rightarrow \neg D) \wedge (\neg D \rightarrow A))$ $\equiv ((\neg A \vee \neg D) \wedge (\neg\neg D \vee A))$ $\equiv ((\neg A \vee \neg D) \wedge (D \vee A))$	$\{\neg A, \neg D\}$ $\{A, D\}$
$(\neg A \rightarrow (B \vee C))$ $\equiv (\neg\neg A \vee (B \vee C))$ $\equiv (A \vee B \vee C)$	$\{A, B, C\}$
$((A \rightarrow E) \wedge (B \vee C \vee F))$ $\equiv ((\neg A \vee E) \wedge (B \vee C \vee F))$	$\{\neg A, E\}$ $\{B, C, F\}$
$(E \rightarrow (F \rightarrow (B \vee C)))$ $\equiv (\neg E \vee (\neg F \vee (B \vee C)))$ $\equiv (\neg E \vee \neg F \vee B \vee C)$	$\{B, C, \neg E, \neg F\}$
$(C \rightarrow G)$ $\equiv (\neg C \vee G)$	$\{\neg C, G\}$
$(G \rightarrow \neg C)$ $\equiv (\neg G \vee \neg C)$	$\{\neg C, \neg G\}$
$\neg(B \wedge \neg C)$ $\equiv (\neg B \vee C)$	$\{\neg B, C\}$

We need to derive the emtpy clause $\Box$ from the following set of clauses Δ:

$$\Delta = \{\{\neg A, \neg D\}, \{A, D\}, \{A, B, C\}, \{\neg A, E\}, \{B, C, F\}, \\ \{B, C, \neg E, \neg F\}, \{\neg C, G\}, \{\neg C, \neg G\}, \{\neg B, C\}\}.$$

One possible derivation:

$K_1 = \{\neg C, G\}$	from Δ
$K_2 = \{\neg C, \neg G\}$	from Δ
$K_3 = \{\neg C\}$	from K_1 and K_2
$K_4 = \{A, B, C\}$	from Δ
$K_5 = \{A, B\}$	from K_3 and K_4
$K_6 = \{C, \neg B\}$	from Δ
$K_7 = \{\neg B\}$	from K_3 and K_6
$K_8 = \{A\}$	from K_5 and K_7
$K_9 = \{\neg A, E\}$	from Δ
$K_{10} = \{E\}$	from K_8 and K_9
$K_{11} = \{B, C, \neg E, \neg F\}$	from Δ
$K_{12} = \{B, C, \neg F\}$	from K_{10} and K_{11}
$K_{13} = \{C, \neg F\}$	from K_7 and K_{12}
$K_{14} = \{\neg F\}$	from K_3 and K_{13}
$K_{15} = \{B, C, F\}$	from Δ
$K_{16} = \{C, F\}$	from K_7 and K_{15}
$K_{17} = \{C\}$	from K_{14} and K_{16}
$K_{18} = \Box$	fromK_3 and K_{17}

With help of the contradiction theorem we can conclude that $KB \models (B \wedge \neg C)$.

Exercise 3.3 (Predicate Logic – Terminology; 2 marks)

Classify the following expressions as *terms*, *ground terms*, *atoms*, *formulas* or *meta language* (statements that are not part of predicate logic itself but statement about the semantics). If for an expression several specifications are correct, please list all of them.

In the expressions a and b are constant symbols, x and y variable symbols, f and g function symbols and P and Q predicate symbols.

(a) $\mathrm{P}(x, y)$

(b) $\mathrm{f}(\mathrm{a}, \mathrm{b})$

(c) $\mathcal{I} \models \mathrm{P}(\mathrm{a}, \mathrm{f}(\mathrm{b}))$

(d) $\mathcal{I}, \alpha \models \mathrm{P}(\mathrm{a}, \mathrm{f}(x))$

(e) $\mathrm{f}(\mathrm{g}(x), \mathrm{b})$

(f) $\mathrm{Q}(x)$ is satisfiable.

(g) $(\exists x \mathrm{P}(x, y) \land \mathrm{Q}(x)) \lor \mathrm{P}(y, x)$

(h) $\forall x(\exists y \mathrm{P}(x, y) \land \mathrm{Q}(x)) \lor \mathrm{P}(x, y)$

(i) $\forall x \forall y(\mathrm{P}(x, y) \land \mathrm{Q}(x) \lor \mathrm{P}(\mathrm{f}(y), x))$

(j) $\mathrm{Q}(x) \lor \mathrm{P}(x, y) \equiv \mathrm{P}(x, y) \lor \mathrm{Q}(x)$

Solution:

- Terms: b, e
- Ground terms: b
- Atoms: a
- Formulas: a, g, h, i
- Statements in meta language: c, d, f, j

Exercise 3.4 (Predicate Logic; 3 Points)
Consider the following predicate logic formula φ with the signature $\langle\{x\}, \{\mathrm{c}\}, \{\mathrm{f}\}, \{\mathrm{P}\}\rangle$.

$$\varphi = (\exists x\, (\mathrm{P}(x) \land \neg\mathrm{P}(\mathrm{f}(x))) \land \forall x\, \neg(\mathrm{f}(x) = \mathrm{c}))$$

Specify a model $\mathcal{I}$ of φ with $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ and $U = \{u_1, u_2, u_3\}$. Prove that $\mathcal{I} \models \varphi$. Why is no variable assignment α required to specify a model of φ?

Solution:
The following interpretation $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ is a model of φ.

$$\begin{aligned} U &= \{u_1, u_2, u_3\} \\ \mathrm{c}^{\mathcal{I}} &= u_1 \\ \mathrm{f}^{\mathcal{I}} &= \{u_1 \mapsto u_2, u_2 \mapsto u_2, u_3 \mapsto u_2\} \\ \mathrm{P}^{\mathcal{I}} &= \{u_3\} \end{aligned}$$

We consider the two conjuncts independently first and assume an arbitrary variable assignment α.

Proof for $\mathcal{I}, \alpha \models \exists x\, (P(x) \land \neg P(f(x)))$:
We consider the variable assignment $\alpha' = \alpha[x := u_3]$. Under this variable assignment, we have $x^{\mathcal{I},\alpha'} = \alpha'(x) = u_3 \in \mathrm{P}^{\mathcal{I}}$ and $\mathrm{f}(x)^{\mathcal{I},\alpha'} = \mathrm{f}^{\mathcal{I}}(\alpha'(x)) = \mathrm{f}^{\mathcal{I}}(u_3) = u_2 \notin \mathrm{P}^{\mathcal{I}}$.
From $x^{\mathcal{I},\alpha'} \in \mathrm{P}^{\mathcal{I}}$ it follows that $\mathcal{I}, \alpha' \models \mathrm{P}(x)$. Analogously, it follows from $\mathrm{f}(x)^{\mathcal{I},\alpha'} \notin \mathrm{P}^{\mathcal{I}}$ that $\mathcal{I}, \alpha' \not\models \mathrm{P}(\mathrm{f}(x))$ and thus $\mathcal{I}, \alpha' \models \neg\mathrm{P}(\mathrm{f}(x))$. From $\mathcal{I}, \alpha' \models \mathrm{P}(x)$ and $\mathcal{I}, \alpha' \models \neg\mathrm{P}(\mathrm{f}(x))$ we can conclude that $\mathcal{I}, \alpha' \models (\mathrm{P}(x) \land \neg\mathrm{P}(\mathrm{f}(x)))$.
Since there is a $u \in U$ (specifically u_3) with $\mathcal{I}, \alpha[x := u] \models (\mathrm{P}(x) \land \neg\mathrm{P}(\mathrm{f}(x)))$ it follows that $\mathcal{I}, \alpha \models \exists x\, (\mathrm{P}(x) \land \neg\mathrm{P}(\mathrm{f}(x)))$.

Proof for $\mathcal{I}, \alpha \models \forall x\, \neg(f(x) = c)$:
For every $u \in U$ we consider the variable assignment $\alpha' = \alpha[x := u]$. No matter what u is, we get $\mathrm{f}(x)^{\mathcal{I},\alpha'} = \mathrm{f}^{\mathcal{I}}(\alpha'(x)) = \mathrm{f}^{\mathcal{I}}(u) = u_2$.
We also know that $\mathrm{c}^{\mathcal{I},\alpha'} = \mathrm{c}^{\mathcal{I}} = u_1 \neq u_2$ and thus $\mathrm{f}(x)^{\mathcal{I},\alpha'} \neq \mathrm{c}^{\mathcal{I},\alpha'}$. This entails $\mathcal{I}, \alpha' \not\models (\mathrm{f}(x) = \mathrm{c})$ and thus $\mathcal{I}, \alpha' \models \neg(\mathrm{f}(x) = \mathrm{c})$.
Since $\mathcal{I}, \alpha[x := u] \models \neg(\mathrm{f}(x) = \mathrm{c})$ is true for all $u \in U$ it follows that $\mathcal{I}, \alpha \models \forall x\, \neg(\mathrm{f}(x) = \mathrm{c})$.

From $\mathcal{I}, \alpha \models \exists x\, (\mathrm{P}(x) \land \neg\mathrm{P}(\mathrm{f}(x)))$ and $\mathcal{I}, \alpha \models \forall x\, \neg(\mathrm{f}(x) = \mathrm{c})$ it directly follows that $\mathcal{I}, \alpha \models \varphi$.
Since all variables are bound, the proof does not depend on the variable assignment α, so it is not required to specify a model.