Theory of Computer Science

G. Röger | University of Basel
Spring Term 2020 | Computer Science

Exercise Sheet 1 — Solutions

Exercise 1.1 (Structural Induction; 3 marks)
We define two functions over binary trees (as presented in the lecture): $height : \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to its height and $leaves : \mathcal{B} \to \mathbb{N}_0$ maps a binary tree $B \in \mathcal{B}$ to the number of its leaves.

- $height(\Box) = 0$
- $height((B_L, \bigcirc, B_R)) = \max(height(B_L), height(B_R)) + 1$
- $leaves(\Box) = 1$
- $leaves((B_L, \bigcirc, B_R)) = leaves(B_L) + leaves(B_R)$

Prove the following property for all binary trees $B \in \mathcal{B}$ by structural induction:

$$leaves(B) \leq 2^{height(B)}.$$

Solution:
We prove the statement by induction over the structure of binary trees.
Induction basis: The property is true for the base case $B = \Box$:

$$leaves(\Box) = 1 = 2^{1-1} = 2^{height(\Box)}.$$

Induction hypothesis:
$leaves(B_L) \leq 2^{height(B_L)}$ and $leaves(B_R) \leq 2^{height(B_R)}$ for binary trees B_L and B_R.

Induction step from B_L and B_R to $B = (B_L, \bigcirc, B_R)$:

$$\begin{aligned}
leaves((B_L, \bigcirc, B_R)) &= leaves(B_L) + leaves(B_R) \\
&\overset{(IV)}{\leq} 2^{height(B_L)} + 2^{height(B_R)} \\
&\leq 2^{\max(height(B_L), height(B_R))} + 2^{\max(height(B_L), height(B_R))} \\
&= 2\left(2^{\max(height(B_L), height(B_R))}\right) \\
&= 2^{\max(height(B_L), height(B_R))+1} \\
&= 2^{height((B_L, \bigcirc, B_R))}
\end{aligned}$$

Exercise 1.2 (Formalization in Propositional Logic; 0.5+0.5+0.5+0.5 marks)
Formalize the following statements as propositional formulas. In order to do so, also define appropriate atomic propositions. Take care to fully parenthesize all formulas.

(a) If it does not rain, it is warm.

Solution:

$(\neg rain \to warm)$

(b) If Bob is going for a swim, then he always eats ice cream and it is not raining.

Solution:

$(BobSwims \to (BobEatsIcecream \land \neg rain))$

(c) Bob is going for a swim exactly if he eats ice cream and it is warm or does not rain.

Solution:

Natural language is ambigous here. The statement can be understood in two ways:
$(\mathit{BobSwims} \leftrightarrow (\mathit{BobEatsIcecream} \wedge (\mathit{warm} \vee \neg \mathit{rain})))$
$(\mathit{BobSwims} \leftrightarrow ((\mathit{BobEatsIcecream} \wedge \mathit{warm}) \vee \neg \mathit{rain}))$

(d) Either the sun is shining or it is raining (but not both).

Solution:

$((\mathit{sun} \vee \mathit{rain}) \wedge \neg(\mathit{sun} \wedge \mathit{rain}))$

Exercise 1.3 (Semantics of Propositional Logic; 2+2 marks)
Consider formula $\varphi = ((A \wedge \neg B) \rightarrow (\neg A \vee \neg C))$ over $\{A, B, C\}$.

(a) Specify a model $\mathcal{I}$ for φ and prove that $\mathcal{I} \models \varphi$.

Solution:

$$\mathcal{I} = \{A \mapsto 0, B \mapsto 0, C \mapsto 0\}$$

Since $\mathcal{I}(A) = 0$, it holds that $\mathcal{I} \not\models A$ and hence $\mathcal{I} \not\models (A \wedge \neg B)$. This implies that $\mathcal{I} \models \neg(A \wedge \neg B)$ and as a result, it holds that $\mathcal{I} \models (\neg(A \wedge \neg B) \vee \psi)$ for arbitrary formulas ψ. In particular, this is true for $(\neg(A \wedge \neg B) \vee (\neg A \vee \neg C))$, which is the formula abbreviated by φ.

(b) Specify an interpretation $\mathcal{I}$ under which ϕ does *not* hold and prove that $\mathcal{I} \not\models \varphi$.

Solution:

$$\mathcal{I} = \{A \mapsto 1, B \mapsto 0, C \mapsto 1\}$$

It is easy to see that $\mathcal{I} \models A$ and $\mathcal{I} \models \neg B$ (because $\mathcal{I} \not\models B$). Hence, it follows that $\mathcal{I} \models (A \wedge \neg B)$ and from this that $\mathcal{I} \not\models \neg(A \wedge \neg B)$ (*).

From $\mathcal{I}(C) = 1$, we get $\mathcal{I} \models C$, implying that $\mathcal{I} \not\models \neg C$. Analogously, $\mathcal{I}(A) = 1$ implies that $\mathcal{I} \not\models \neg A$ and together it follows that $\mathcal{I} \not\models (\neg A \vee \neg C)$ (**).

From (*) and (**), we conclude that $\mathcal{I} \not\models (\neg(A \wedge \neg B) \vee (\neg A \vee \neg C)$ and thus $\mathcal{I} \not\models \varphi$.

Exercise 1.4 (Semantics of Propositional Logic; 1 marks)
Let φ and ψ be propositional formulas over the same set A of atomic propositions and let $\mathcal{I}$ be an interpretation for A. Prove that $\mathcal{I} \models (\varphi \rightarrow \psi)$ iff $\mathcal{I} \not\models \varphi$ or $\mathcal{I} \models \psi$.

Solution:

Expression $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$, which is true under $\mathcal{I}$ iff $\mathcal{I} \models \neg\varphi$ or $\mathcal{I} \models \psi$. The statement follows from $\mathcal{I} \models \neg\varphi$ iff $\mathcal{I} \not\models \varphi$.