Theory of Computer Science

F1. LOOP-Computability

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Overview: Course

contents of this course:

A. background ✓
- ▷ mathematical foundations and proof techniques

B. logic ✓
- ▷ How can knowledge be represented?
  How can reasoning be automated?

C. automata theory and formal languages ✓
- ▷ What is a computation?

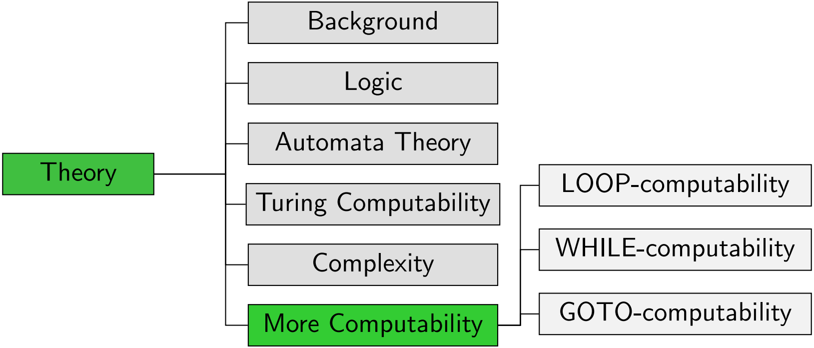
D. Turing computability ✓
- ▷ What can be computed at all?

E. complexity theory ✓
- ▷ What can be computed efficiently?

F. more computability theory
- ▷ Other models of computability

Introduction

Course Overview



Theory

Background

Logic

Automata Theory

Turing Computability

Complexity

More Computability

LOOP-computability

WHILE-computability

GOTO-computability

Formal Models of Computation: LOOP/WHILE/GOTO

Formal Models of Computation

- Turing machines
- LOOP, WHILE and GOTO programs
- (primitive recursive and μ-recursive functions)

In this and the following chapter we get to know three simple models of computation (programming languages) and compare their power to Turing machines:

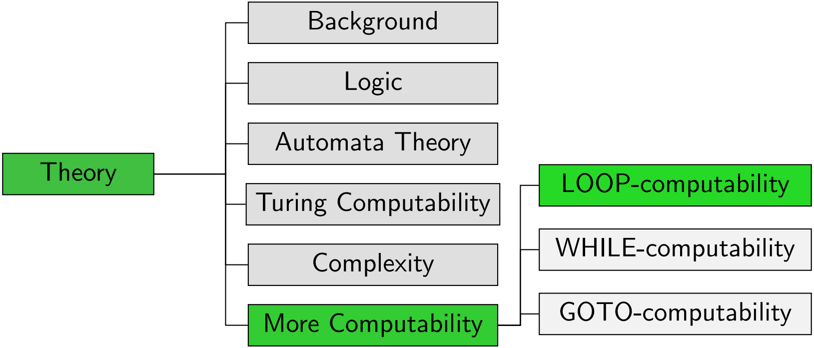
- LOOP programs $\rightsquigarrow$ today
- WHILE programs $\rightsquigarrow$ F2
- GOTO programs $\rightsquigarrow$ F3

LOOP, WHILE and GOTO Programs: Basic Concepts

- LOOP, WHILE and GOTO programs are structured like programs in (simple) “traditional” programming languages
- use finitely many variables from the set $\{x_0, x_1, x_2, \dots\}$ that can take on values in $\mathbb{N}_0$
- differ from each other in the allowed “statements”

LOOP Programs

Course Overview


Theory

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LOOP Programs: Syntax

Definition (LOOP Program)

LOOP programs are inductively defined as follows:

- $x_i := x_j + c$ is a LOOP program
  for every $i, j, c \in \mathbb{N}_0$ (addition)
- $x_i := x_j - c$ is a LOOP program
  for every $i, j, c \in \mathbb{N}_0$ (modified subtraction)
- If P_1 and P_2 are LOOP programs,
  then so is $P_1;P_2$ (composition)
- If P is a LOOP program, then so is
  LOOP x_i DO P END for every $i \in \mathbb{N}_0$ (LOOP loop)

German: LOOP-Programm, Addition,
modifizierte Subtraktion, Komposition, LOOP-Schleife

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

A LOOP program computes a k-ary function $f : \mathbb{N}_0^k \to \mathbb{N}_0$. The computation of $f(n_1, \dots, n_k)$ works as follows:

1. Initially, the variables $x_1, \dots, x_k$ hold the values $n_1, \dots, n_k$. All other variables hold the value 0.
2. During computation, the program modifies the variables as described on the following slides.
3. The result of the computation ($f(n_1, \dots, n_k)$) is the value of x_0 after the execution of the program.

German: P berechnet f

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of $x_i := x_j + c$:

- The variable x_i is assigned the current value of x_j plus c.
- All other variables retain their value.

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of $x_i := x_j - c$:

- The variable x_i is assigned the current value of x_j minus c if this value is non-negative.
- Otherwise x_i is assigned the value 0.
- All other variables retain their value.

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of P_1; P_2:

- First, execute P_1.
  Then, execute P_2 (on the modified variable values).

LOOP Programs: Semantics

Definition (Semantics of LOOP Programs)

effect of LOOP x_i DO P END:

- Let m be the value of variable x_i at the start of execution.
- The program P is executed m times in sequence.

LOOP-Computable Functions

Definition (LOOP-Computable)

A function $f : \mathbb{N}_0^k \rightarrow_{\mathrm{p}} \mathbb{N}_0$ is called LOOP-computable if a LOOP program that computes f exists.

German: f ist LOOP-berechenbar

Note: non-total functions are never LOOP-computable.
(Why not?)

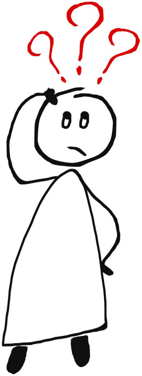
LOOP Programs: Example

Example (LOOP program for $f(x_1, x_2)$)

```
LOOP $x_1$ DO
  LOOP $x_2$ DO
    $x_0 := x_0 + 1$
  END
END
```

Which (binary) function does this program compute?

Questions



Questions?

Syntactic Sugar

Syntactic Sugar or Essential Feature?

- We investigate the power of programming languages and other computation formalisms.
- Rich language features help when writing complex programs.
- Minimalistic formalisms are useful for proving statements over all programs.

$\rightsquigarrow$ conflict of interest!

Idea:

- Use minimalistic core for proofs.
- Use syntactic sugar when writing programs.

German: syntaktischer Zucker

Example: Syntactic Sugar

Example (syntactic sugar)

We propose five new syntax constructs (with the obvious semantics):

- $x_i := x_j$ for $i, j \in \mathbb{N}_0$
- $x_i := c$ for $i, c \in \mathbb{N}_0$
- $x_i := x_j + x_k$ for $i, j, k \in \mathbb{N}_0$
- IF $x_i \neq 0$ THEN P END for $i \in \mathbb{N}_0$
- IF $x_i = c$ THEN P END for $i, c \in \mathbb{N}_0$

Can we simulate these with the existing constructs?

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := x_j$ for $i, j \in \mathbb{N}_0$

Simulation with existing constructs?

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := x_j$ for $i, j \in \mathbb{N}_0$

Simple abbreviation for $x_i := x_j + 0$.

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := c$ for $i, c \in \mathbb{N}_0$

Simulation with existing constructs?

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := c$ for $i, c \in \mathbb{N}_0$

Simple abbreviation for $x_i := x_j + c$,
where x_j is a fresh variable, i.e., an otherwise unused variable that is not an input variable.
(Thus x_j must always have the value 0 in all executions.)

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := x_j + x_k$ for $i, j, k \in \mathbb{N}_0$

Simulation with existing constructs?

Example: Syntactic Sugar

Example (syntactic sugar)

$x_i := x_j + x_k$ for $i, j, k \in \mathbb{N}_0$

Abbreviation for:

```
$x_i := x_j$;
LOOP $x_k$ DO
  $x_i := x_i + 1$
END
```

Analogously we will also use the following:

- $x_i := x_j - x_k$
- $x_i := x_j + x_k - c - x_m + d$
- etc.

Example: Syntactic Sugar

Example (syntactic sugar)

IF $x_i \neq 0$ THEN P END for $i \in \mathbb{N}_0$

Simulation with existing constructs?

Example: Syntactic Sugar

Example (syntactic sugar)

IF $x_i \neq 0$ THEN P END for $i \in \mathbb{N}_0$

Abbreviation for:

```
$x_j := 0$;
LOOP $x_i$ DO
  $x_j := 1$
END;
LOOP $x_j$ DO
  $P$
END
```

where x_j is a fresh variable.

Example: Syntactic Sugar

Example (syntactic sugar)

IF $x_i = c$ THEN P END for $i, c \in \mathbb{N}_0$

Simulation with existing constructs?

Example: Syntactic Sugar

Example (syntactic sugar)

IF $x_i = c$ THEN P END for $i, c \in \mathbb{N}_0$

Abbreviation for:

```
$x_j := 1$;
$x_k := x_i - c$;
IF $x_k \neq 0$ THEN $x_j := 0$ END;
$x_k := c - x_i$;
IF $x_k \neq 0$ THEN $x_j := 0$ END;
IF $x_j \neq 0$ THEN
    $P$
END
```

where x_j and x_k are fresh variables.

Can We Be More Minimalistic?

- We see that some common structural elements such as IF statements are unnecessary because they are syntactic sugar.
- Can we make LOOP programs even more minimalistic than in our definition?

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Simplification 1

Instead of $x_i := x_j + c$ and $x_i := x_j - c$ it suffices to only allow the constructs

- $x_i := x_j$,
- $x_i := x_i + 1$ and
- $x_i := x_i - 1$.

Why?

Can We Be More Minimalistic?

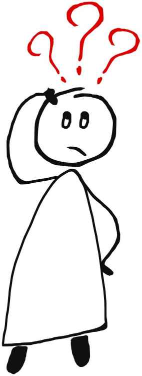
- We see that some common structural elements such as IF statements are unnecessary because they are syntactic sugar.
- Can we make LOOP programs even more minimalistic than in our definition?

Simplification 2

The construct $x_i := x_j$ can be omitted because it can be simulated with other constructs:

```
LOOP $x_i$ DO
  $x_i := x_i - 1$
END;
LOOP $x_j$ DO
  $x_i := x_i + 1$
END
```

Questions



Questions?

Summary

Summary

LOOP programs

- new model of computation for numerical functions
- closer to typical programming languages than Turing machines