Theory of Computer Science

E3. Proving NP-Completeness

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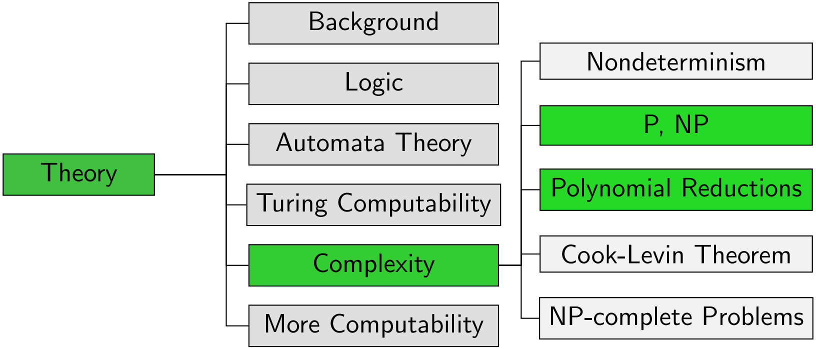
Course Overview



E3.1 Overview

Proving NP-Completeness by Reduction

- Suppose we know one NP-complete problem (we will use satisfiability of propositional logic formulas).
- With its help, we can then prove quite easily that further problems are NP-complete.

Theorem (Proving NP-Completeness by Reduction)

Let A and B be problems such that:

- *A is NP-hard, and*
- *$A \leq_{\mathrm{p}} B$.*

Then B is also NP-hard.
If furthermore $B \in \mathrm{NP}$, then B is NP-complete.

Proving NP-Completeness by Reduction: Proof

Proof.

First part: We must show $X \leq_{\mathrm{p}} B$ for all $X \in \mathrm{NP}$.

From $X \leq_{\mathrm{p}} A$ (because A is NP-hard) and $A \leq_{\mathrm{p}} B$ (by prerequisite), this follows due to the transitivity of $\leq_{\mathrm{p}}$.

Second part: follows directly by definition of NP-completeness. □

NP-Complete Problems

- There are thousands of known NP-complete problems.
- An extensive catalog of NP-complete problems from many areas of computer science is contained in:

  Michael R. Garey and David S. Johnson:
  Computers and Intractability —
  A Guide to the Theory of NP-Completeness
  W. H. Freeman, 1979.

- In the remaining chapters, we get to know some of these problems.

Overview of the Reductions

$$\begin{tikzcd}
 & \text{SAT} \arrow[d] & \\
 & \text{3SAT} \arrow[dl] \arrow[d] \arrow[dr] & \\
\textsc{Clique} \arrow[d] & \textsc{DirHamiltonCycle} \arrow[d] & \textsc{SubsetSum} \arrow[d] \\
\textsc{IndSet} \arrow[d] & \textsc{HamiltonCycle} \arrow[d] & \textsc{Partition} \arrow[d] \\
\textsc{VertexCover} & \text{TSP} & \textsc{BinPacking}
\end{tikzcd}$$

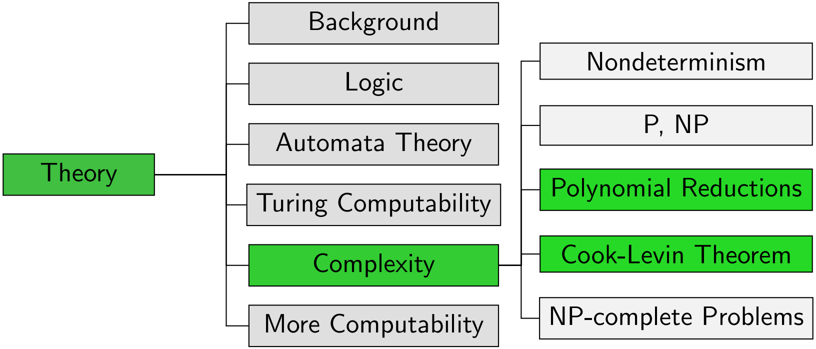
What Do We Have to Do?

- We want to show the NP-completeness of these 11 problems.
- We first show that SAT is NP-complete.
- Then it is sufficient to show
  - that polynomial reductions exist for all edges in the figure (and thus all problems are NP-hard)
  - and that the problems are all in NP.

(It would be sufficient to show membership in NP only for the leaves in the figure. But membership is so easy to show that this would not save any work.)

E3.2 Cook-Levin Theorem

Course Overview



SAT is NP-complete

Definition (SAT)

The problem SAT (satisfiability) is defined as follows:

Given: a propositional logic formula φ

Question: Is φ satisfiable?

Theorem (Cook, 1971; Levin, 1973)

SAT *is NP-complete.*

Proof.

SAT $\in$ NP: guess and check.

SAT is NP-hard: somewhat more complicated (to be continued)

...

NP-hardness of SAT (1)

Proof (continued).

We must show: $A \leq_{\mathsf{p}} \mathrm{SAT}$ for all $A \in \mathsf{NP}$.

Let A be an arbitrary problem in NP.

We have to find a polynomial reduction of A to SAT,
i. e., a function f computable in polynomial time
such that for every input word w over the alphabet of A:

$w \in A$ iff $f(w)$ is a satisfiable propositional formula. ...

NP-hardness of SAT (2)

Proof (continued).

Because $A \in \mathsf{NP}$, there is an NTM M and a polynomial p such that M accepts the problem A in time p.

Idea: construct a formula that encodes the possible configurations which M can reach in time $p(|w|)$ on input w and that is satisfiable if and only if an end configuration can be reached in this time. ...

NP-hardness of SAT (3)

Proof (continued).

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \Box, E \rangle$ be an NTM for A,
and let p be a polynomial bounding the computation time of M.
Without loss of generality, $p(n) \geq n$ for all n.

Let $w = w_1 \dots w_n \in \Sigma^*$ be the input for M.

We number the tape positions with integers (positive and negative) such that the TM head initially is on position 1.

Observation: within $p(n)$ computation steps the TM head can only reach positions in the set
$Pos = \{-p(n)+1, -p(n)+2, \dots, -1, 0, 1, \dots, p(n)+1\}$.

Instead of infinitely many tape positions, we now only need to consider these (polynomially many!) positions. …

NP-hardness of SAT (4)

Proof (continued).

We can encode configurations of M by specifying:

- what the current state of M is
- on which position in Pos the TM head is located
- which symbols from Γ the tape contains at positions Pos

$\rightsquigarrow$ can be encoded by propositional variables

To encode a full computation (rather than just one configuration), we need copies of these variables for each computation step.

We only need to consider the computation steps $\mathit{Steps} = \{0, 1, \dots, p(n)\}$ because M should accept within $p(n)$ steps. ...

NP-hardness of SAT (5)

Proof (continued).

Use the following propositional variables in formula $f(w)$:

- $\mathit{state}_{t,q}$ $(t \in \mathit{Steps}, q \in Q)$
  $\rightsquigarrow$ encodes the state of the NTM in the t-th configuration
- $\mathit{head}_{t,i}$ $(t \in \mathit{Steps}, i \in \mathit{Pos})$
  $\rightsquigarrow$ encodes the head position in the t-th configuration
- $\mathit{tape}_{t,i,a}$ $(t \in \mathit{Steps}, i \in \mathit{Pos}, a \in \Gamma)$
  $\rightsquigarrow$ encodes the tape content in the t-th configuration

Construct $f(w)$ such that every satisfying interpretation

- describes a sequence of TM configurations
- that begins with the start configuration,
- reaches an accepting configuration
- and follows the TM rules in δ

...

NP-hardness of SAT (6)

Proof (continued).

Auxiliary formula:

$$\mathit{oneof}\, X := \left(\bigvee_{x \in X} x \right) \land \neg \left(\bigvee_{x \in X} \bigvee_{y \in X \setminus \{x\}} (x \land y) \right)$$

Auxiliary notation:

The symbol $\perp$ stands for an arbitrary unsatisfiable formula (e.g., $(A \land \neg A)$, where A is an arbitrary proposition). ...

NP-hardness of SAT (7)

Proof (continued).

1. describe the configurations of the TM:

$$\mathit{Valid} := \bigwedge_{t \in \mathit{Steps}} \Big(\mathit{oneof}\,\{\mathit{state}_{t,q} \mid q \in Q\} \land \mathit{oneof}\,\{\mathit{head}_{t,i} \mid i \in \mathit{Pos}\} \land \bigwedge_{i \in \mathit{Pos}} \mathit{oneof}\,\{\mathit{tape}_{t,i,a} \mid a \in \Gamma\} \Big)$$

...

NP-hardness of SAT (8)

Proof (continued).

2. begin in the start configuration

$$\mathit{Init} := \mathit{state}_{0,q_0} \wedge \mathrm{head}_{0,1} \wedge \bigwedge_{i=1}^{n} \mathit{tape}_{0,i,w_i} \wedge \bigwedge_{i \in \mathit{Pos} \setminus \{1,\dots,n\}} \mathit{tape}_{0,i,\square}$$

...

NP-hardness of SAT (9)

Proof (continued).

3. reach an accepting configuration

$$Accept := \bigvee_{t \in Steps} \bigvee_{q_e \in E} state_{t,q_e}$$

...

NP-hardness of SAT (10)

Proof (continued).

4. follow the rules in δ:

$$\mathit{Trans} := \bigwedge_{t \in \mathit{Steps}} \left(\bigvee_{q_{\mathrm{e}} \in E} \mathit{state}_{t,q_{\mathrm{e}}} \vee \bigvee_{R \in \delta} \mathit{Rule}_{t,R} \right)$$

where. . . . . .

NP-hardness of SAT (11)

Proof (continued).

4. follow the rules in δ (continued):

$$
\begin{aligned}
&\mathit{Rule}_{t,\langle\langle q,a\rangle,\langle q',a',D\rangle\rangle} := \\
&\qquad \mathit{state}_{t,q} \land \mathit{state}_{t+1,q'} \land \\
&\qquad \bigwedge_{i \in \mathit{Pos}} \left(\mathit{head}_{t,i} \to \mathit{tape}_{t,i,a} \land \mathit{head}_{t+1,i+D} \land \mathit{tape}_{t+1,i,a'}\right) \land \\
&\qquad \bigwedge_{i \in \mathit{Pos}} \bigwedge_{a'' \in \Gamma} \left(\neg \mathit{head}_{t,i} \land \mathit{tape}_{t,i,a''} \to \mathit{tape}_{t+1,i,a''}\right)
\end{aligned}
$$

- For D, interpret L $\rightsquigarrow -1$, N $\rightsquigarrow 0$, R $\rightsquigarrow +1$.
- special case: *tape* and *head* variables with a tape index $i + D$ outside of *Pos* are replaced by $\perp$; likewise all variables with a time index outside of *Steps*.

...

NP-hardness of SAT (12)

Proof (continued).

Putting the pieces together:

Set $f(w) := \mathit{Valid} \wedge \mathit{Init} \wedge \mathit{Accept} \wedge \mathit{Trans}$.

- $f(w)$ can be constructed in time polynomial in $|w|$.
- $w \in A$ iff M accepts w in $p(|w|)$ steps
  iff $f(w)$ is satisfiable
  iff $f(w) \in \mathrm{SAT}$

$\rightsquigarrow$ $A \leq_{\mathrm{p}} \mathrm{SAT}$

Since $A \in \mathrm{NP}$ was arbitrary, this is true for every $A \in \mathrm{NP}$.
Hence SAT is NP-hard and thus also NP-complete. $\square$

E3.3 3SAT

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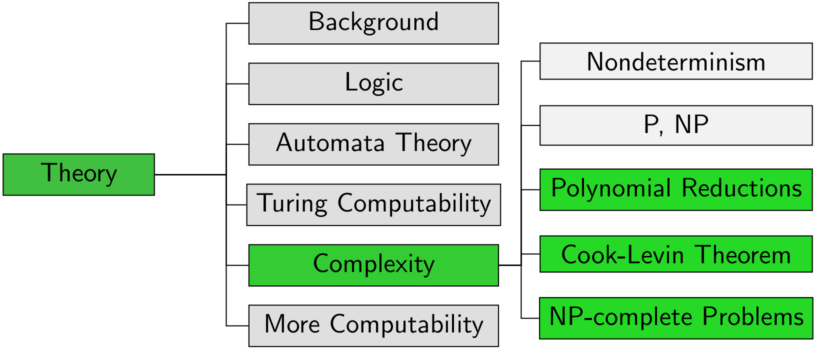

SAT $\leq_p$ 3SAT

$$\begin{tikzcd}
 & \text{SAT} \arrow[d] & \\
 & \text{3SAT} \arrow[dl] \arrow[d] \arrow[dr] & \\
\textsc{Clique} \arrow[d] & \textsc{DirHamiltonCycle} \arrow[d] & \textsc{SubsetSum} \arrow[d] \\
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\end{tikzcd}$$

SAT and 3SAT

Definition (Reminder: SAT)

The problem SAT (satisfiability) is defined as follows:

Given: a propositional logic formula φ

Question: Is φ satisfiable?

Definition (3SAT)

The problem 3SAT is defined as follows:

Given: a propositional logic formula φ in conjunctive normal form with at most three literals per clause

Question: Is φ satisfiable?

3SAT is NP-Complete (1)

Theorem (3SAT is NP-Complete)

3SAT *is NP-complete.*

3SAT is NP-Complete (2)

Proof.

$\mathrm{3SAT} \in \mathsf{NP}$: guess and check.

$\mathrm{3SAT}$ is NP-hard: We show $\mathrm{SAT} \leq_{\mathsf{p}} \mathrm{3SAT}$.

- Let φ be the given input for SAT. Let $\mathit{Sub}(\varphi)$ denote the set of subformulas of φ, including φ itself.
- For all $\psi \in \mathit{Sub}(\varphi)$, we introduce a new proposition X_ψ.
- For each new proposition X_ψ, define the following auxiliary formula χ_ψ:
  - If $\psi = A$ for an atom A: $\chi_\psi = (X_\psi \leftrightarrow A)$
  - If $\psi = \neg\psi'$: $\chi_\psi = (X_\psi \leftrightarrow \neg X_{\psi'})$
  - If $\psi = (\psi' \wedge \psi'')$: $\chi_\psi = (X_\psi \leftrightarrow (X_{\psi'} \wedge X_{\psi''}))$
  - If $\psi = (\psi' \vee \psi'')$: $\chi_\psi = (X_\psi \leftrightarrow (X_{\psi'} \vee X_{\psi''}))$

...

3SAT is NP-Complete (3)

Proof (continued).

- Consider the conjunction of all these auxiliary formulas, $\chi_{\text{all}} := \bigwedge_{\psi \in \mathit{Sub}(\varphi)} \chi_\psi$.
- Every interpretation $\mathcal{I}$ of the original variables can be extended to a model $\mathcal{I}'$ of χ_{all} in exactly one way: for each $\psi \in \mathit{Sub}(\varphi)$, set $\mathcal{I}'(X_\psi) = 1$ iff $\mathcal{I} \models \psi$.
- It follows that φ is satisfiable iff $(\chi_{\text{all}} \wedge X_\varphi)$ is satisfiable.
- This formula can be computed in linear time.
- It can also be converted to 3-CNF in linear time because it is the conjunction of constant-size parts involving at most three variables each. (Each part can be converted to 3-CNF independently.)
- Hence, this describes a polynomial-time reduction.

□

Restricted 3SAT

Note: 3SAT remains NP-complete if we also require that

- every clause contains exactly three literals and
- a clause may not contain the same literal twice

Idea:

- remove duplicated literals from each clause.
- add new variables: X, Y, Z
- add new clauses: $(X \lor Y \lor Z)$, $(X \lor Y \lor \neg Z)$, $(X \lor \neg Y \lor Z)$, $(\neg X \lor Y \lor Z)$, $(X \lor \neg Y \lor \neg Z)$, $(\neg X \lor Y \lor \neg Z)$, $(\neg X \lor \neg Y \lor Z)$

$\leadsto$ satisfied if and only if X, Y, Z are all true

- fill up clauses with fewer than three literals with $\neg X$ and if necessary additionally with $\neg Y$

E3.4 Summary

Summary

- Thousands of important problems are NP-complete.
- The satisfiability problem of propositional logic (SAT) is NP-complete.
- Proof idea for NP-hardness:
  - Every problem in NP can be solved by an NTM in polynomial time $p(|w|)$ for input w.
  - Given a word w, construct a propositional logic formula φ that encodes the computation steps of the NTM on input w.
  - Construct φ so that it is satisfiable if and only if there is an accepting computation of length $p(|w|)$.
- Usually (as seen for $3\mathrm{SAT}$), the easiest way to show that another problem is NP-complete is to
  - show that it is in NP with a guess-and-check algorithm, and
  - polynomially reduce a known NP-complete to it.