

Theory of Computer Science

E2. P, NP and Polynomial Reductions

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E2.1 P and NP

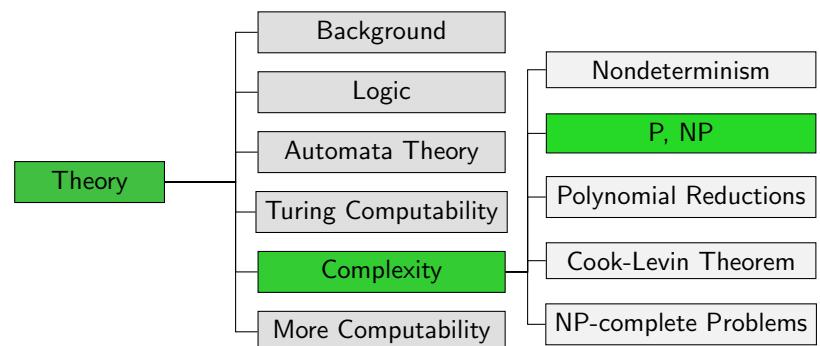
E2.2 Polynomial Reductions

E2.3 NP-Hardness and NP-Completeness

E2.4 Summary

E2.1 P and NP

Course Overview



Accepting a Word in Time n

Definition (Accepting a Word in Time n)

Let M be a DTM or NTM with input alphabet Σ , $w \in \Sigma^*$ a word and $n \in \mathbb{N}_0$.

M accepts w in time n if there is a sequence of configurations c_0, \dots, c_k with $k \leq n$, where:

- ▶ c_0 is the start configuration for w ,
- ▶ $c_0 \vdash c_1 \vdash \dots \vdash c_k$, and
- ▶ c_k is an end configuration.

German: M akzeptiert w in Zeit n

Accepting a Language in Time f

Definition (Accepting a Language in Time f)

Let M be a DTM or NTM with input alphabet Σ , $L \subseteq \Sigma^*$ a language and $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ a function.

M accepts L in time f if:

- ① for all words $w \in L$: M accepts w in time $f(|w|)$
- ② for all words $w \notin L$: M does not accept w

German: M akzeptiert L in Zeit f

P and NP

Definition (P and NP)

P is the set of all languages L for which a **DTM** M and a **polynomial** p exist such that M accepts L in time p .

NP is the set of all languages L for which an **NTM** M and a **polynomial** p exist such that M accepts L in time p .

P and NP: Remarks

- ▶ Sets of languages like P and NP that are defined in terms of computation time of TMs (or other computation models) are called **complexity classes**.
- ▶ We know that $P \subseteq NP$. ([Why?](#))
- ▶ Whether the converse is also true is an open question: this is the famous **P-NP problem**.

German: Komplexitätsklassen, P-NP-Problem

Example: $\text{DIRHAMILTONCYCLE} \in \text{NP}$

Example ($\text{DIRHAMILTONCYCLE} \in \text{NP}$)

The nondeterministic algorithm of Chapter E1 solves the problem and can be implemented on an NTM in polynomial time.

- ▶ Is $\text{DIRHAMILTONCYCLE} \in \text{P}$ also true?
- ▶ The answer is unknown.
- ▶ So far, only exponential deterministic algorithms for the problem are known.

Simulation of NTMs with DTM

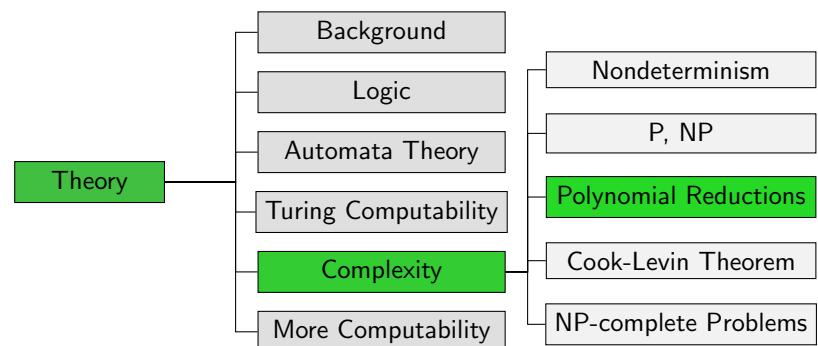
- ▶ Unlike DTM, NTMs are not a **realistic** computation model: they cannot be directly implemented on computers.
- ▶ But NTMs can be **simulated** by systematically trying all computation paths, e. g., with a **breadth-first search**.

More specifically:

- ▶ Let M be an NTM that accepts language L in time f , where $f(n) \geq n$ for all $n \in \mathbb{N}_0$.
- ▶ Then we can specify a DTM M' that accepts L in time f' , where $f'(n) = 2^{O(f(n))}$.
- ▶ **without proof**
(cf. "Introduction to the Theory of Computation" by Michael Sipser (3rd edition), Theorem 7.11)

E2.2 Polynomial Reductions

Course Overview



Polynomial Reductions: Idea

- ▶ **Reductions** are a common and powerful concept in computer science. We know them from Part D.
- ▶ The basic idea is that we solve a new problem by **reducing** it to a known problem.
- ▶ In complexity theory we want to use reductions that allow us to prove statements of the following kind:

$$\text{Problem } A \text{ can be solved efficiently} \\ \text{if problem } B \text{ can be solved efficiently.}$$
- ▶ For this, we need a reduction from A to B that can be computed efficiently itself (otherwise it would be useless for efficiently solving A).

Polynomial Reductions: Remarks

- ▶ Polynomial reductions are also called **Karp reductions** (after Richard Karp, who wrote a famous paper describing many such reductions in 1972).
- ▶ In practice, of course we do not have to specify a DTM for f : it just has to be clear that f can be computed in **polynomial time** by a **deterministic algorithm**.

Polynomial Reductions

Definition (Polynomial Reduction)

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be decision problems.

We say that A can be **polynomially reduced to B** , written $A \leq_p B$, if there is a function $f : \Sigma^* \rightarrow \Gamma^*$ such that:

- ▶ f can be computed in **polynomial time** by a **DTM**
 - ▶ i. e., there is a polynomial p and a DTM M such that M computes $f(w)$ in at most $p(|w|)$ steps given input $w \in \Sigma^*$
- ▶ f reduces A to B
 - ▶ i. e., for all $w \in \Sigma^*$: $w \in A$ iff $f(w) \in B$

f is called a **polynomial reduction** from A to B

German: A polnomiell auf B reduzierbar,
polnomielle Reduktion von A auf B

Polynomial Reductions: Remarks

Polynomial Reductions: Example (1)

Definition (HAMILTONCYCLE)

HAMILTONCYCLE is the following decision problem:

- ▶ **Given:** undirected graph $G = \langle V, E \rangle$
- ▶ **Question:** Does G contain a Hamilton cycle?

Reminder:

Definition (Hamilton Cycle)

A **Hamilton cycle** of G is a sequence of vertices in V , $\pi = \langle v_0, \dots, v_n \rangle$, with the following properties:

- ▶ π is a path: there is an edge from v_i to v_{i+1} for all $0 \leq i < n$
- ▶ π is a cycle: $v_0 = v_n$
- ▶ π is simple: $v_i \neq v_j$ for all $i \neq j$ with $i, j < n$
- ▶ π is Hamiltonian: all nodes of V are included in π

Polynomial Reductions: Example (2)

Definition (TSP)

TSP (traveling salesperson problem) is the following decision problem:

- ▶ **Given:** finite set $S \neq \emptyset$ of cities, symmetric cost function $cost : S \times S \rightarrow \mathbb{N}_0$, cost bound $K \in \mathbb{N}_0$
- ▶ **Question:** Is there a tour with total cost at most K , i. e., a permutation $\langle s_1, \dots, s_n \rangle$ of the cities with $\sum_{i=1}^{n-1} cost(s_i, s_{i+1}) + cost(s_n, s_1) \leq K$?

German: Problem der/des Handlungsreisenden

Polynomial Reductions: Example (3)

Theorem ($\text{HAMILTONCYCLE} \leq_p \text{TSP}$)

$\text{HAMILTONCYCLE} \leq_p \text{TSP}$.

Proof.

↔ blackboard



Properties of Polynomial Reductions (1)

Theorem (Properties of Polynomial Reductions)

Let A , B and C decision problems.

- ① If $A \leq_p B$ and $B \in P$, then $A \in P$.
- ② If $A \leq_p B$ and $B \in NP$, then $A \in NP$.
- ③ If $A \leq_p B$ and $A \notin P$, then $B \notin P$.
- ④ If $A \leq_p B$ and $A \notin NP$, then $B \notin NP$.
- ⑤ If $A \leq_p B$ and $B \leq_p C$, then $A \leq_p C$.

Properties of Polynomial Reductions (2)

Proof.

for 1.:

We must show that there is a DTM accepting A in polynomial time.

We know:

- ▶ There is a DTM M_B that accepts B in time p , where p is a polynomial.
- ▶ There is a DTM M_f that computes a reduction from A to B in time q , where q is a polynomial.

...

Properties of Polynomial Reductions (3)

Proof (continued).

Consider the machine M that first behaves like M_f , and then (after M_f stops) behaves like M_B on the output of M_f .

M accepts A :

- ▶ M behaves on input w as M_B does on input $f(w)$, so it accepts w if and only if $f(w) \in B$.
- ▶ Because f is a reduction, $w \in A$ iff $f(w) \in B$.

...

Properties of Polynomial Reductions (5)

Proof (continued).

for 2.:

analogous to 1., only that M_B and M are NTMs

of 3.+4.:

equivalent formulations of 1.+2. (contraposition)

of 5.:

Let $A \leq_p B$ with reduction f and $B \leq_p C$ with reduction g .

Then $g \circ f$ is a reduction of A to C .

The computation time of the two computations in sequence is polynomial by the same argument used in the proof for 1. □

Properties of Polynomial Reductions (4)

Proof (continued).

Computation time of M on input w :

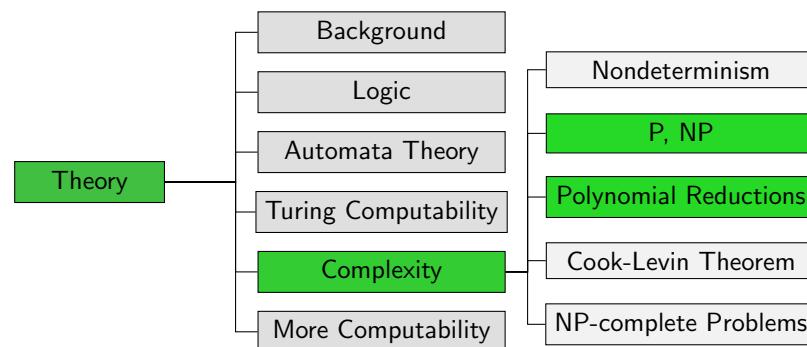
- ▶ first M_f runs on input w : $\leq q(|w|)$ steps
- ▶ then M_B runs on input $f(w)$: $\leq p(|f(w)|)$ steps
- ▶ $|f(w)| \leq |w| + q(|w|)$ because in $q(|w|)$ steps, M_f can write at most $q(|w|)$ additional symbols onto the tape
- ~~> total computation time $\leq q(|w|) + p(|f(w)|)$
 $\leq q(|w|) + p(|w| + q(|w|))$
- ~~> this is polynomial in $|w|$ ~~> $A \in P$.

...

Properties of Polynomial Reductions (5)

E2.3 NP-Hardness and NP-Completeness

Course Overview



NP-Hardness and NP-Completeness

Definition (NP-Hard, NP-Complete)

Let B be a decision problem.

B is called **NP-hard** if $A \leq_p B$ for **all** problems $A \in \text{NP}$.

B is called **NP-complete** if $B \in \text{NP}$ and B is NP-hard.

German: NP-hart (selten: NP-schwer), NP-vollständig

NP-Complete Problems: Meaning

- ▶ NP-hard problems are “at least as difficult” as all problems in NP.
- ▶ NP-complete problems are “the most difficult” problems in NP: **all** problems in NP can be reduced to them.
- ▶ If $A \in \text{P}$ for **any** NP-complete problem, then $\text{P} = \text{NP}$. ([Why?](#))
- ▶ That means that either there are efficient algorithms for **all** NP-complete problems or for **none** of them.
- ▶ **Do NP-complete problems actually exist?**

E2.4 Summary

Summary

- ▶ **P**: languages accepted by **DTMs** in polynomial time
- ▶ **NP**: languages accepted by **NTMs** in polynomial time
- ▶ **polynomial reductions**: $A \leq_p B$ if
 - there is a total function f computable **in polynomial time**,
 - such that for all words w : $w \in A$ iff $f(w) \in B$
- ▶ $A \leq_p B$ implies that A is "**at most as difficult**" as B
- ▶ polynomial reductions are **transitive**
- ▶ **NP-hard** problems B : $A \leq_p B$ for **all** $A \in \text{NP}$
- ▶ **NP-complete** problems B : $B \in \text{NP}$ and B is NP-hard