

# Theory of Computer Science

## C7. Context-Sensitive and Type-0 Languages: Turing Machines

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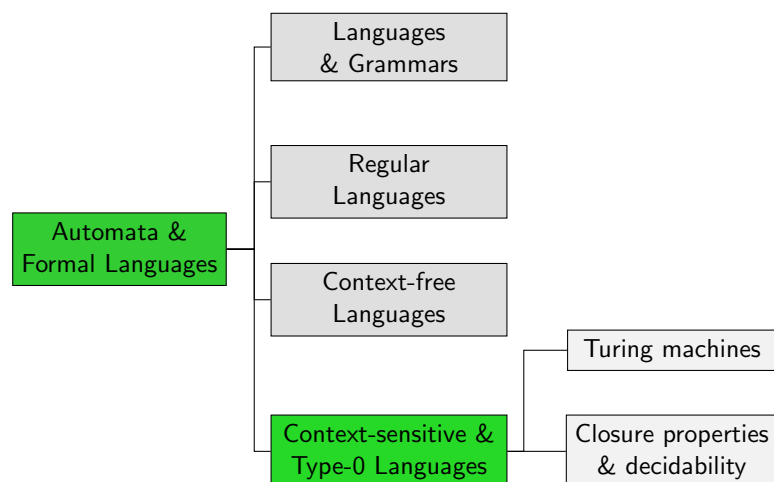
April 3, 2019 — C7. Context-Sensitive and Type-0 Languages: Turing Machines

## C7.1 Context-Sensitive and General Grammars

## C7.2 Turing Machines

## C7.3 Summary

## Overview



# C7.1 Context-Sensitive and General Grammars

## Repetition: (Context-Sensitive) Grammars

### Definition (Grammar)

A **grammar** is a 4-tuple  $\langle \Sigma, V, P, S \rangle$  with:

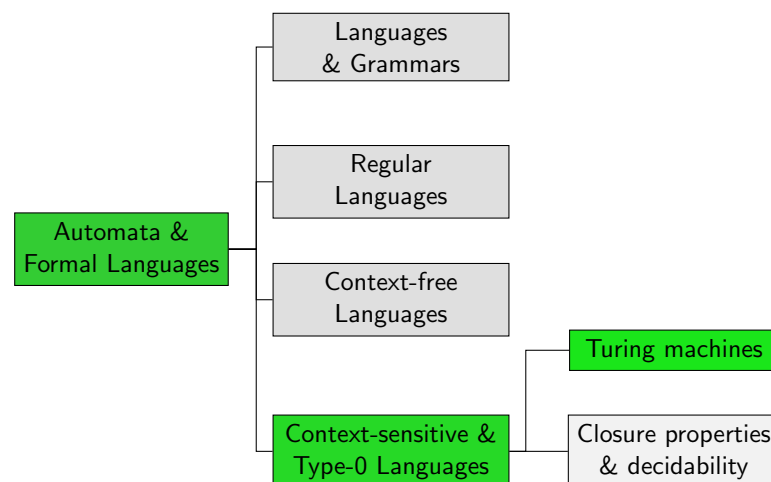
- ▶  $\Sigma$  finite alphabet of terminal symbols
- ▶  $V$  finite set of variables (with  $V \cap \Sigma = \emptyset$ )
- ▶  $P \subseteq (V \cup \Sigma)^+ \times (V \cup \Sigma)^*$  finite set of rules
- ▶  $S \in V$  start variable

Type 0 and type 1:

- ▶ **Every grammar is type 0.**
- ▶ A grammar is **context-sensitive (type-1)** if all rules  $w_1 \rightarrow w_2$  satisfy  $|w_1| \leq |w_2|$ .
  - ▶ **only exception:**  $S \rightarrow \varepsilon$  is allowed for the start symbol  $S$  if  $S$  never occurs on a right-hand side.

## C7.2 Turing Machines

## Overview



## Automata for Type-1 and Type-0 Languages?



Finite automata accept exactly the regular languages, push-down automata exactly the context-free languages. Are there automata models for context-sensitive and type-0 languages?

**Yes!**  $\rightsquigarrow$  **Turing machines**  
**German:** Turingmaschinen

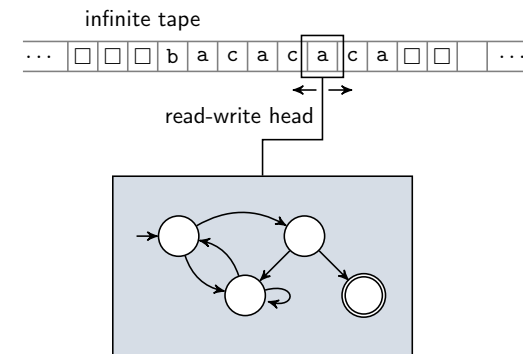
## Alan Turing (1912–1954)



Picture courtesy of Jon Callas /  
wikimedia commons

- ▶ British logician, mathematician, cryptanalyst and computer scientist
- ▶ most important work (for us):  
On Computable Numbers, with an Application to the Entscheidungsproblem  
↪ Turing machines
- ▶ collaboration on Enigma decryption
- ▶ conviction due to homosexuality; pardoned by Elizabeth II in Dec. 2013
- ▶ Turing award most important science award in computer science

## Turing Machines: Conceptually



## Nondeterministic Turing Machine: Definition

### Definition (Nondeterministic Turing Machine)

A nondeterministic Turing machine (NTM) is given by a 7-tuple  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$  with:

- ▶  $Q$  finite non-empty set of states
- ▶  $\Sigma \neq \emptyset$  finite input alphabet
- ▶  $\Gamma \supset \Sigma$  finite tape alphabet
- ▶  $\delta : (Q \setminus E) \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R, N\})$  transition function
- ▶  $q_0 \in Q$  start state
- ▶  $\square \in \Gamma \setminus \Sigma$  blank symbol
- ▶  $E \subseteq Q$  end states

German: Turingmaschine, Zustände, Eingabealphabet, Bandalphabet, Übergangsfunktion, Startzustand, Blank-Symbol, Endzustände

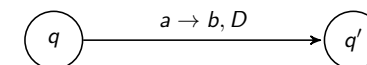
## Turing Machine: Transition Function

Let  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$  be an NTM.

What is the Intuitive Meaning of the Transition Function  $\delta$ ?

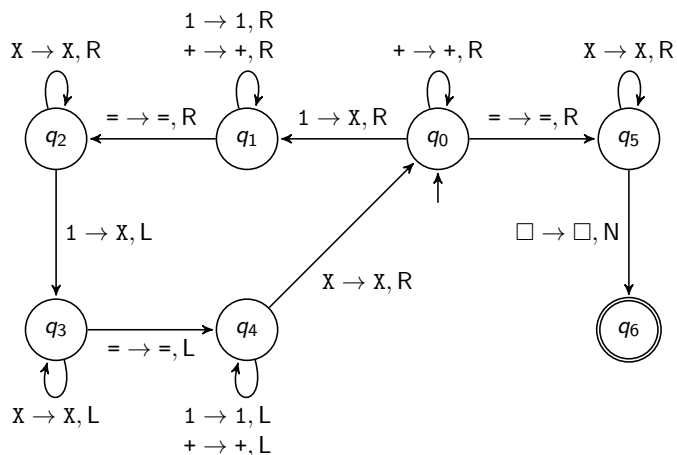
$\langle q', b, D \rangle \in \delta(q, a)$ :

- ▶ If  $M$  is in state  $q$  and reads  $a$ , then
- ▶  $M$  can transition to state  $q'$  in the next step,
- ▶ replacing  $a$  with  $b$ ,
- ▶ and moving the head in direction  $D \in \{L, R, N\}$ , where:
  - ▶ **L**: one step to the left,
  - ▶ **R**: one step to the right,
  - ▶ **N**: neutral (no) movement.



### Nondeterministic Turing Machine: Example

$$M = \langle \{q_0, q_1, \dots, q_6\}, \{1, +, =\}, \{1, +, =, X, \square\}, \delta, q_0, \square, \{q_6\} \rangle$$



### Turing Machine: Configuration

#### Definition (Configuration of a Turing Machine)

A **configuration** of a Turing machine  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$  is given by a triple  $c \in \Gamma^* \times Q \times \Gamma^+$ .

German: Konfiguration

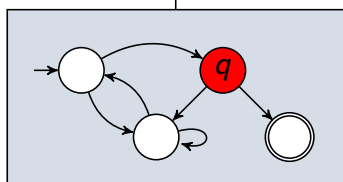
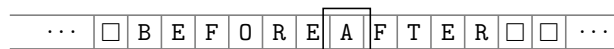
Configuration  $\langle w_1, q, w_2 \rangle$  intuitively means that

- ▶ the non-empty or already visited part of the tape contains the word  $w_1 w_2$ ,
- ▶ the read-write head is on the first symbol of  $w_2$ , and
- ▶ the TM is in state  $q$ .

### Turing Machine Configurations: Example

#### Example

configuration  $\langle \square \text{BEFORE}, q, \text{AFTER} \square \square \rangle$ .



### Turing Machine: Step

#### Definition (Transition/Step of a Turing Machine)

An NTM  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$  can transition from configuration  $c$  to configuration  $c'$  in one step ( $c \vdash_M c'$ ) according to the following rules:

- ▶  $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_m, q', cb_2 \dots b_n \rangle$  if  $\langle q', c, N \rangle \in \delta(q, b_1)$ ,  $m \geq 0$ ,  $n \geq 1$
- ▶  $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_{m-1}, q', a_m cb_2 \dots b_n \rangle$  if  $\langle q', c, L \rangle \in \delta(q, b_1)$ ,  $m \geq 1$ ,  $n \geq 1$
- ▶  $\langle \epsilon, q, b_1 \dots b_n \rangle \vdash_M \langle \epsilon, q', \square cb_2 \dots b_n \rangle$  if  $\langle q', c, L \rangle \in \delta(q, b_1)$ ,  $n \geq 1$
- ▶  $\langle a_1 \dots a_m, q, b_1 \dots b_n \rangle \vdash_M \langle a_1 \dots a_m c, q', b_2 \dots b_n \rangle$  if  $\langle q', c, R \rangle \in \delta(q, b_1)$ ,  $m \geq 0$ ,  $n \geq 2$
- ▶  $\langle a_1 \dots a_m, q, b_1 \rangle \vdash_M \langle a_1 \dots a_m c, q', \square \rangle$  if  $\langle q', c, R \rangle \in \delta(q, b_1)$ ,  $m \geq 0$

## Turing Machines: Reachability of Configurations

### Definition (Reachable Configuration)

Configuration  $c'$  is **reachable** from configuration  $c$  in NTM  $M$  ( $c \vdash_M^* c'$ ) if there are configurations  $c_0, \dots, c_n$  ( $n \geq 0$ ) where

- ▶  $c_0 = c$ ,
- ▶  $c_i \vdash_M c_{i+1}$  for all  $i \in \{0, \dots, n-1\}$ , and
- ▶  $c_n = c'$ .

German:  $c'$  ist in  $M$  von  $c$  erreichbar

## Turing Machines: Recognized Words

### Definition (Recognized Word of a Turing Machine)

NTM  $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E \rangle$  **recognizes the word**  $w = a_1 \dots a_n$  iff an **accepting configuration** (where  $M$  is in an end state) is reachable from the **start configuration**:

$$M \text{ recognizes } w \text{ iff } \langle \varepsilon, q_0, w \rangle \vdash_M^* \langle w_1, q, w_2 \rangle \\ \text{for some } q \in E, w_1 \in \Gamma^*, w_2 \in \Gamma^+$$

**special case:** for  $w = \varepsilon$  the start configuration is  $\langle \varepsilon, q_0, \square \rangle$  rather than  $\langle \varepsilon, q_0, \varepsilon \rangle$

German:  $M$  erkennt  $w$ , akzeptierende Konfiguration, Startkonfiguration

example: blackboard

## Turing Machines: Accepted Language

### Definition (Accepted Language of an NTM)

Let  $M$  be an NTM with input alphabet  $\Sigma$ .

The **language accepted by  $M$**  is defined as

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M \text{ recognizes } w\}.$$

German: erkannte Sprache

example: blackboard

## Exercise

Specify the state diagram of an NTM that accepts language

$$L = \{a^n b^n c^n \mid n \geq 1\}.$$

Feel free to solve this together with your neighbour.

## C7.3 Summary

## Summary

- ▶ Turing machines only have finitely many states but an **unbounded tape** as “memory”.
- ▶ Alan Turing proposed them as a mathematical model for arbitrary algorithmic computations.
- ▶ In this role, we will revisit them in the parts on computability and complexity theory.