Theory of Computer Science

C4. Regular Languages: Minimal Automata, Closure Properties and Decidability

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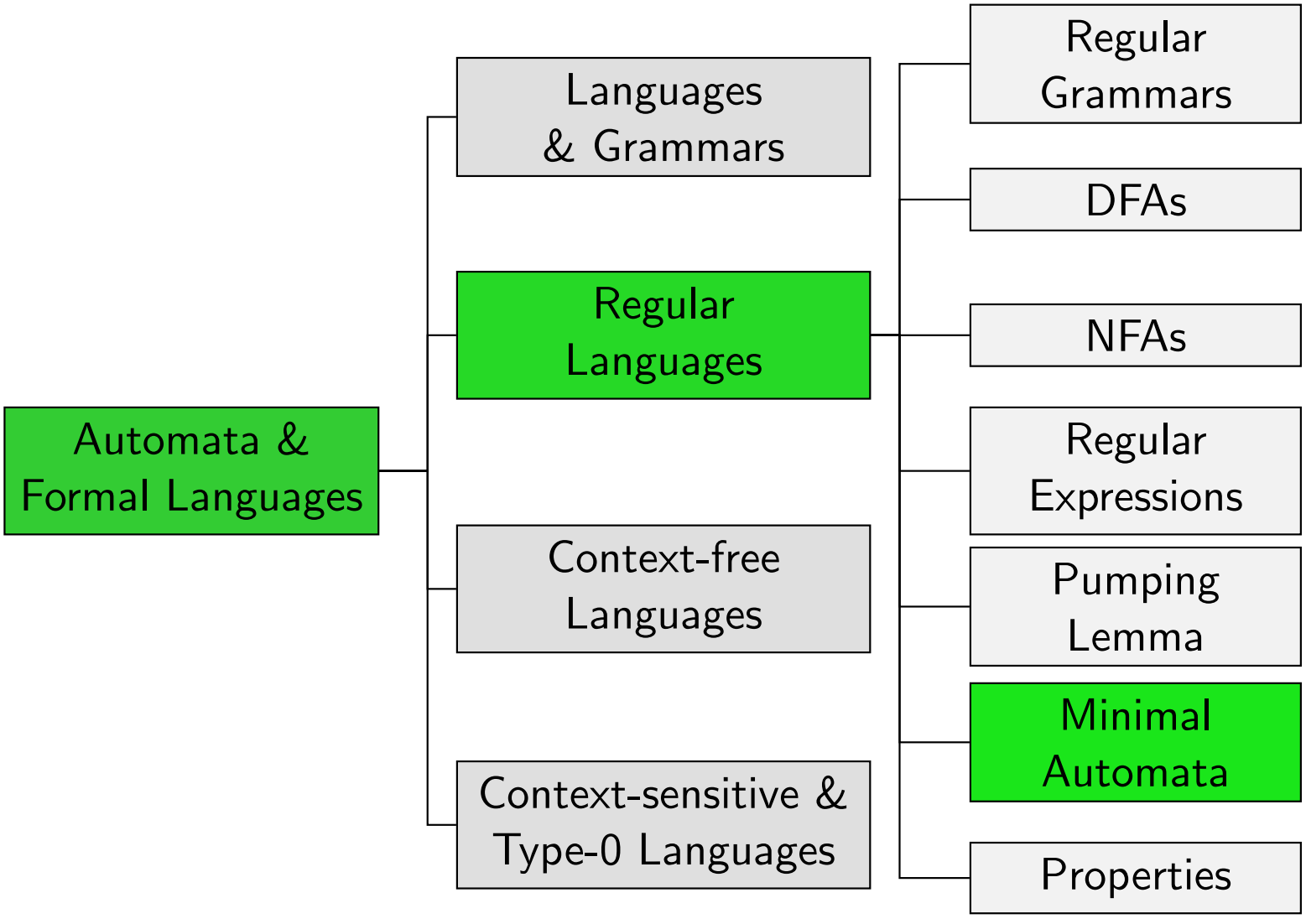
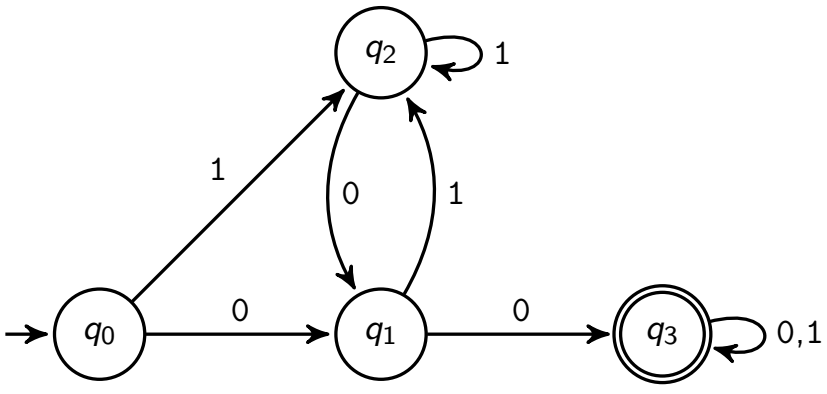
Overview

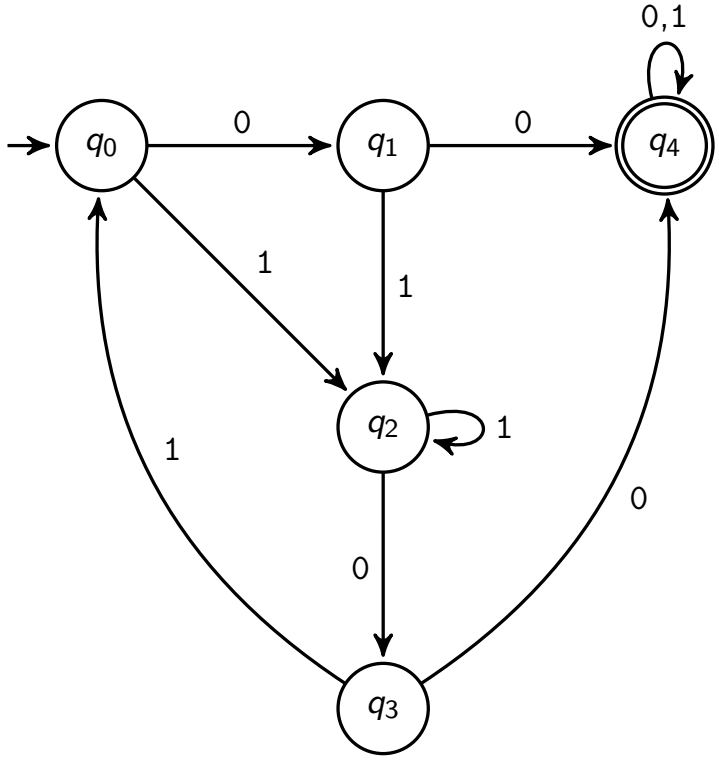

Automata & Formal Languages
Languages & Grammars
Regular Languages
Context-free Languages
Context-sensitive & Type-0 Languages
Regular Grammars
DFAs
NFAs
Regular Expressions
Pumping Lemma
Minimal Automata
Properties

Minimal Automata

Example

The following DFAs accept the same language:





Question: What is the smallest DFA that accepts this language?

Minimal Automaton: Definition

Definition

A minimal automaton for a regular language L is a DFA $M = \langle Q, \Sigma, \delta, q_0, E \rangle$ with $\mathcal{L}(M) = L$ and a minimal number of states.

This means there is no DFA $M' = \langle Q', \Sigma, \delta', q_0', E' \rangle$ with $\mathcal{L}(M) = \mathcal{L}(M')$ and $|Q'| < |Q|$.

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How to find a minimal automaton?

Idea:

- Start with any DFA that accepts the language.
- Merge states from which exactly the same words lead to an end state.

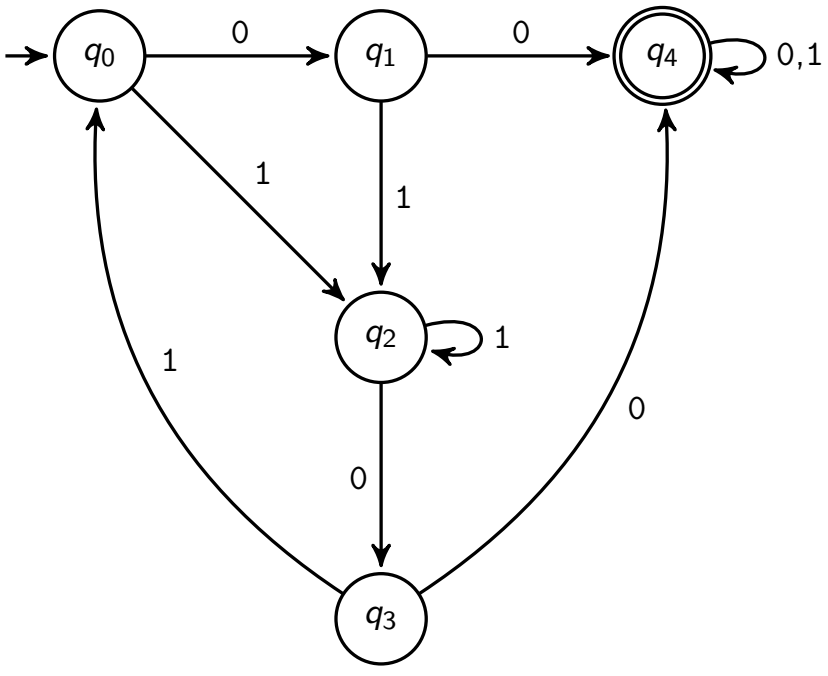
Minimal Automaton: Algorithm

Input: DFA M
(without states that are unreachable from the start state)

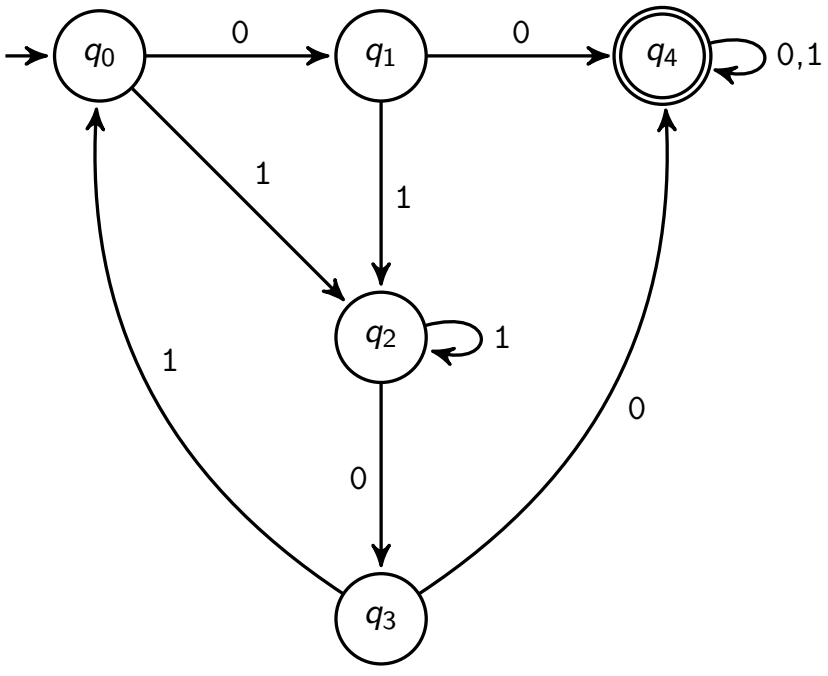
Output: list of states that have to be merged
to obtain an equivalent minimal automaton

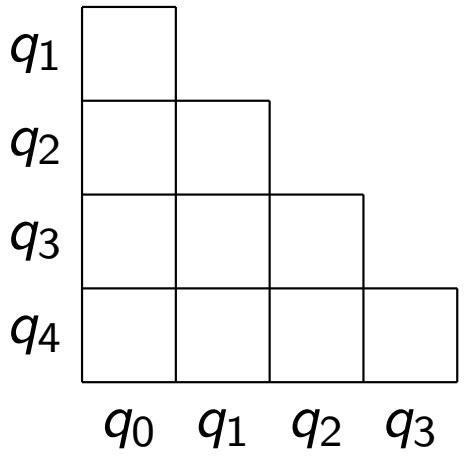
1. Create table of all pairs of states $\{q, q'\}$ with $q \neq q'$.
2. Mark all pairs $\{q, q'\}$ with $q \in E$ and $q' \notin E$.
3. If there is an unmarked pair $\{q, q'\}$ where $\{\delta(q, a), \delta(q', a)\}$ for some $a \in \Sigma$ is already marked, then also mark $\{q, q'\}$.
4. Repeat the last step until there are no more changes.
5. All states in pairs that are still unmarked can be merged into one state.

Minimal Automaton: Example



Minimal Automaton: Example





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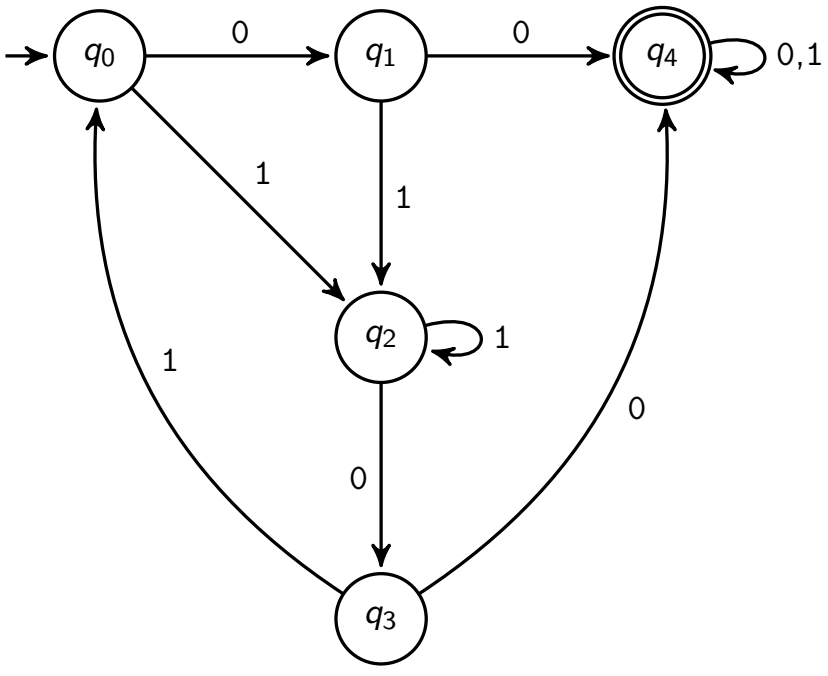
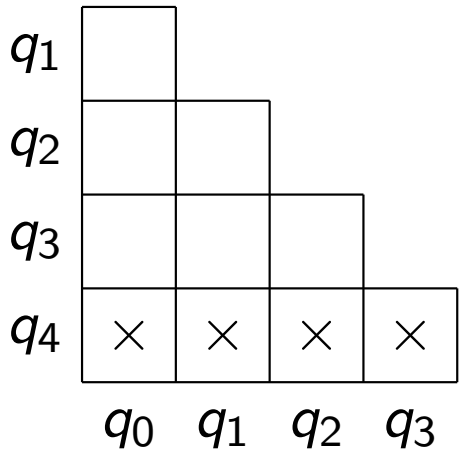
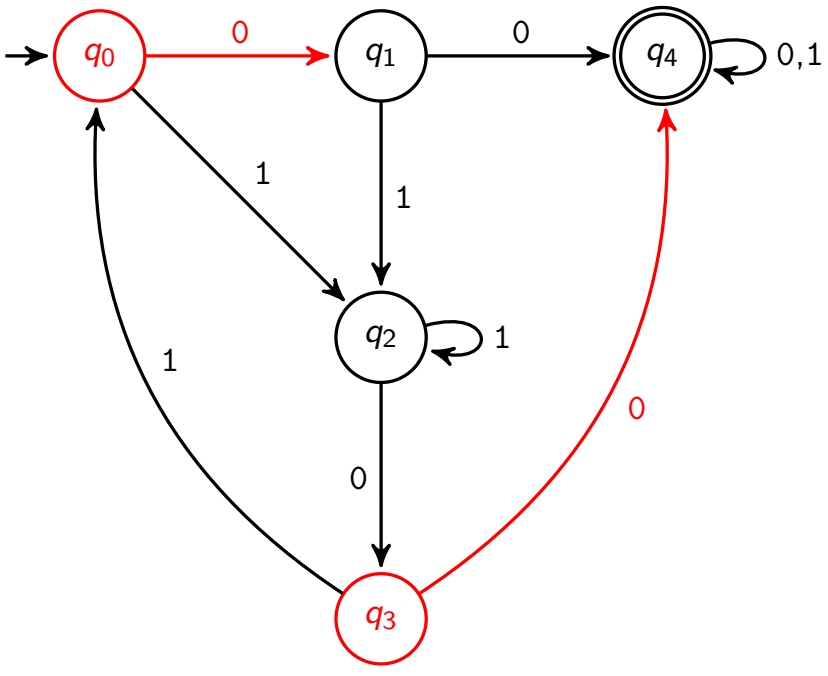
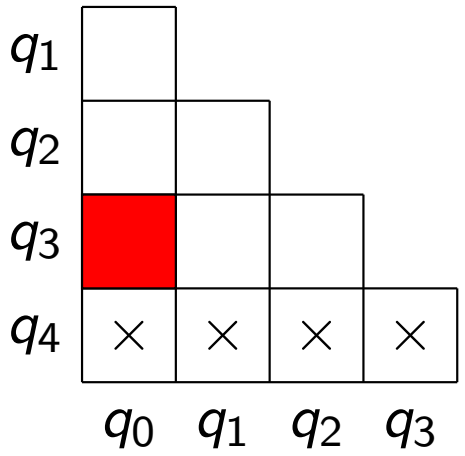




Minimal Automaton: Example





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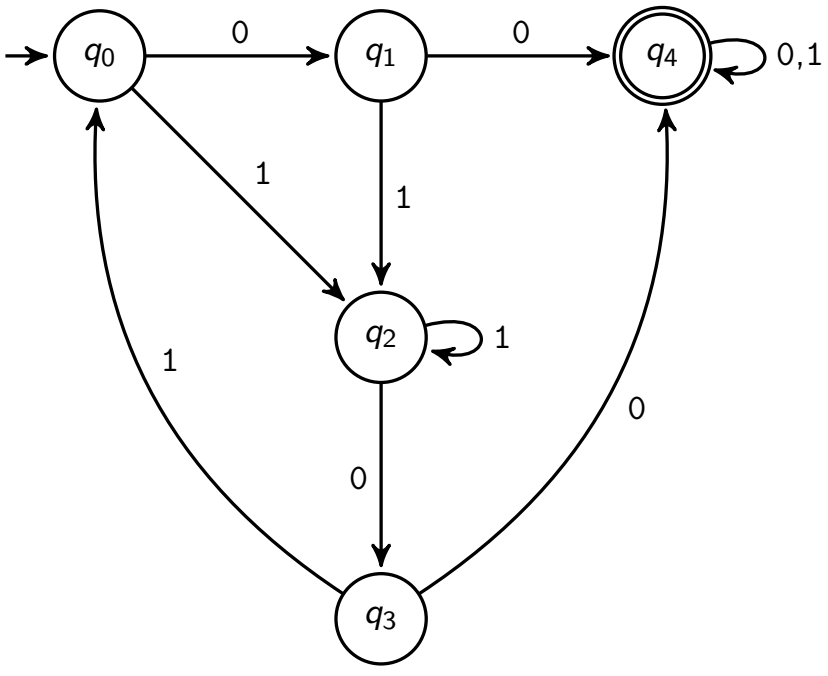

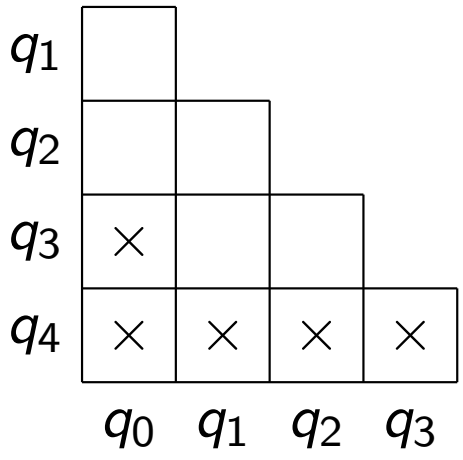

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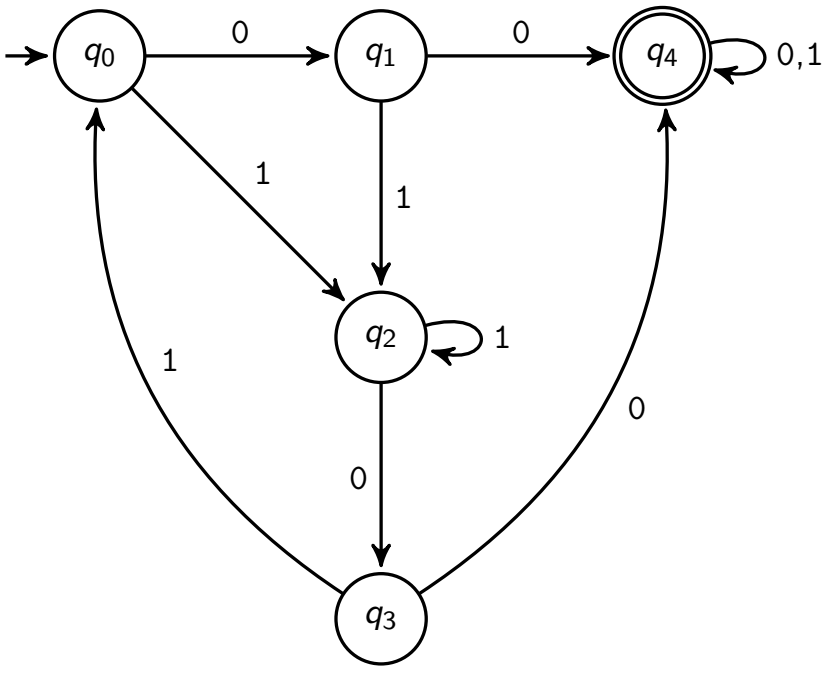
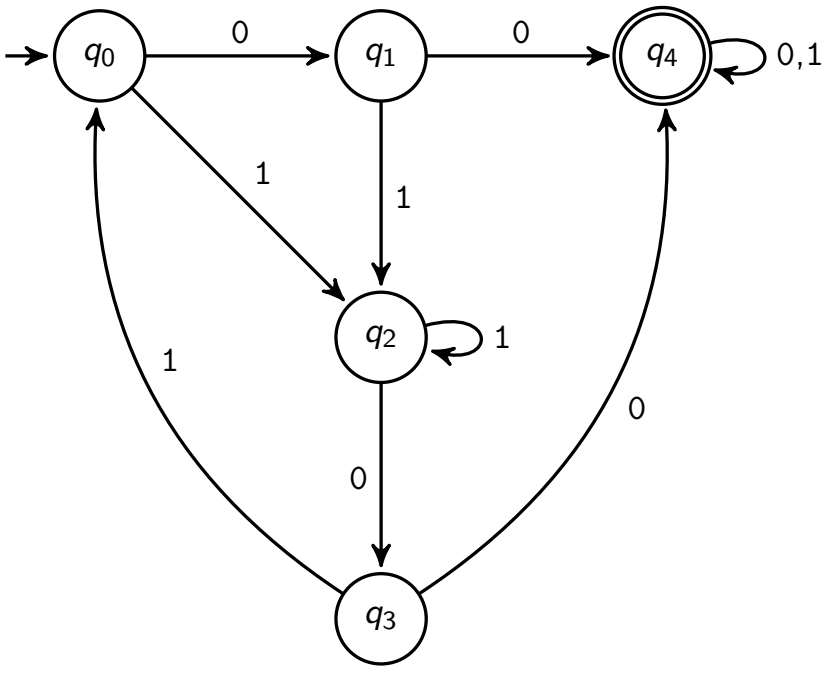


q_1	×			
q_2		×		
q_3	×		×	
q_4	×	×	×	×
	q_0	q_1	q_2	q_3

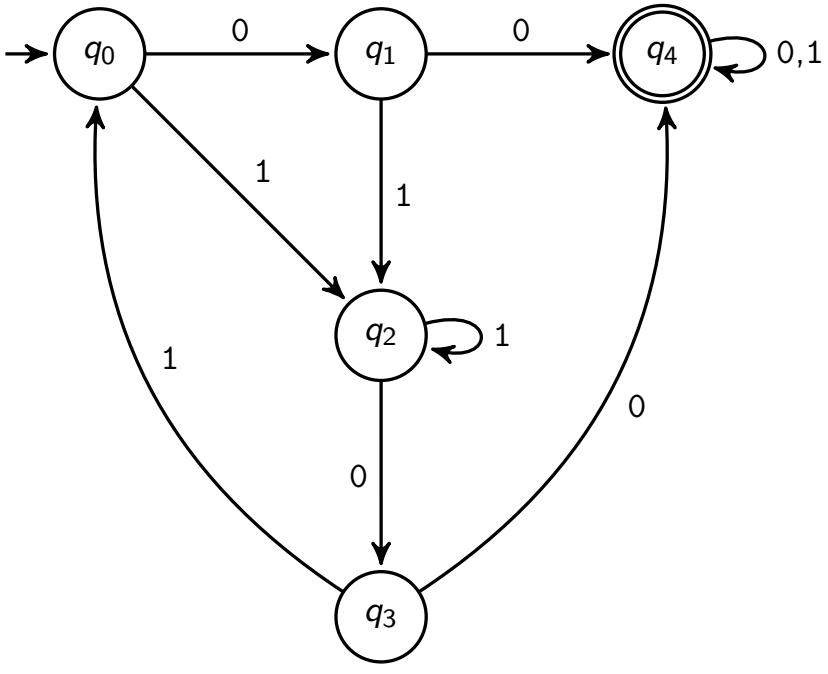
Minimal Automaton: Example



q_1	×			
q_2		×		
q_3	×		×	
q_4	×	×	×	×
	q_0	q_1	q_2	q_3

States q_0, q_2 and q_1, q_3 can be merged into one state each.

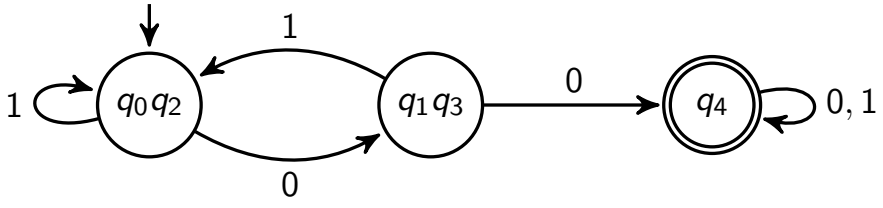
Minimal Automaton: Example



q_1	×			
q_2		×		
q_3	×		×	
q_4	×	×	×	×
	q_0	q_1	q_2	q_3

States q_0, q_2 and q_1, q_3 can be merged into one state each.

Result:



Computation and Uniqueness of Minimal Automata

Theorem

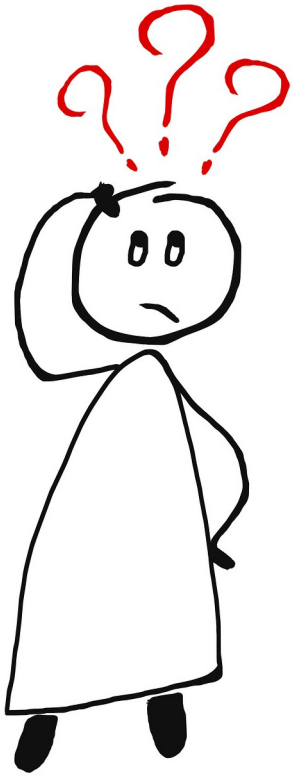
The algorithm described on the previous slides produces a minimal automaton for the language accepted by the given input DFA.

Theorem

All minimal automata for a language L are unique up to isomorphism (i.e., renaming of states).

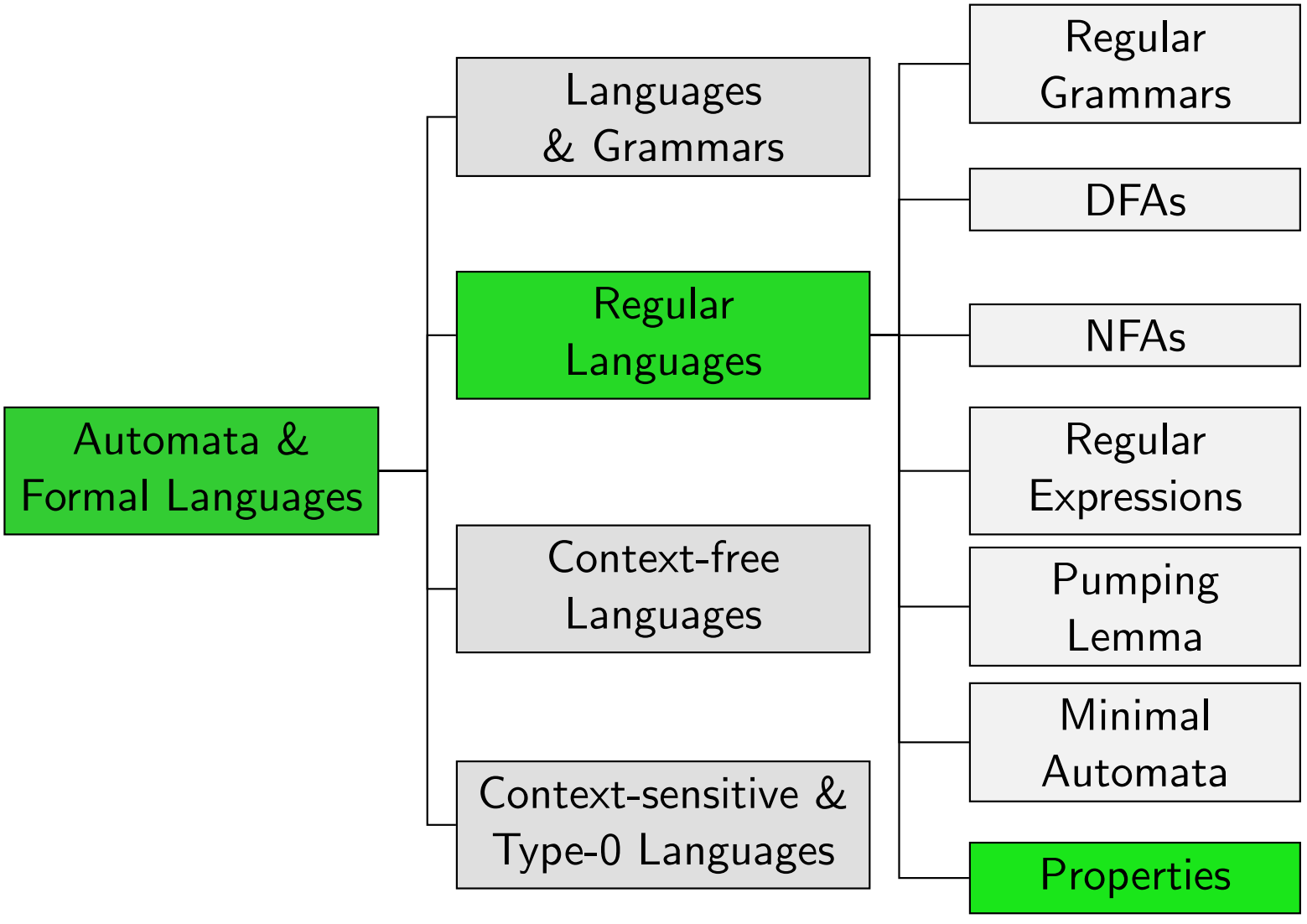
Without proof.

Questions



Questions?

Overview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Closure Properties

Closure Properties

How can you combine regular languages in a way to get another regular language as a result?

Picture courtesy of stockimages / FreeDigitalPhotos.net

Closure Properties: Operations

Let L and L' be regular languages over Σ and Σ', respectively.

We consider the following operations:

- union $L \cup L' = \{w \mid w \in L \text{ or } w \in L'\}$ over $\Sigma \cup \Sigma'$
- intersection $L \cap L' = \{w \mid w \in L \text{ and } w \in L'\}$ over $\Sigma \cap \Sigma'$
- complement $\bar{L} = \{w \in \Sigma^* \mid w \notin L\}$ over Σ
- product $LL' = \{uv \mid u \in L \text{ and } v \in L'\}$ over $\Sigma \cup \Sigma'$
  - special case: $L^n = L^{n-1}L$, where $L^0 = \{\varepsilon\}$
- star $L^* = \bigcup_{k \geq 0} L^k$ over Σ

German: Abschlusseigenschaften, Vereinigung, Schnitt, Komplement, Produkt, Stern

Closure Properties

Definition (Closure)

Let $\mathcal{K}$ be a class of languages.

Then $\mathcal{K}$ is closed...

- ...under union if $L, L' \in \mathcal{K}$ implies $L \cup L' \in \mathcal{K}$
- ...under intersection if $L, L' \in \mathcal{K}$ implies $L \cap L' \in \mathcal{K}$
- ...under complement if $L \in \mathcal{K}$ implies $\bar{L} \in \mathcal{K}$
- ...under product if $L, L' \in \mathcal{K}$ implies $LL' \in \mathcal{K}$
- ...under star if $L \in \mathcal{K}$ implies $L^* \in \mathcal{K}$

German: Abgeschlossenheit, $\mathcal{K}$ ist abgeschlossen unter Vereinigung (Schnitt, Komplement, Produkt, Stern)

Clourse Properties of Regular Languages

Theorem

The regular languages are closed under:

- *union*
- *intersection*
- *complement*
- *product*
- *star*

Closure Properties

Proof.

Closure under union, product, and star follows because for regular expressions α and β, the expressions $(\alpha|\beta)$, $(\alpha\beta)$ and (α^*) describe the corresponding languages.

German: Kreuzproduktautomat

Closure Properties

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Complement: Let $M = \langle Q, \Sigma, \delta, q_0, E \rangle$ be a DFA with $\mathcal{L}(M) = L$. Then $M' = \langle Q, \Sigma, \delta, q_0, Q \setminus E \rangle$ is a DFA with $\mathcal{L}(M') = \bar{L}$.

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Intersection: Let $M_1 = \langle Q_1, \Sigma_1, \delta_1, q_{01}, E_1\rangle$ and $M_2 = \langle Q_2, \Sigma_2, \delta_2, q_{02}, E_2\rangle$ be DFAs. The product automaton

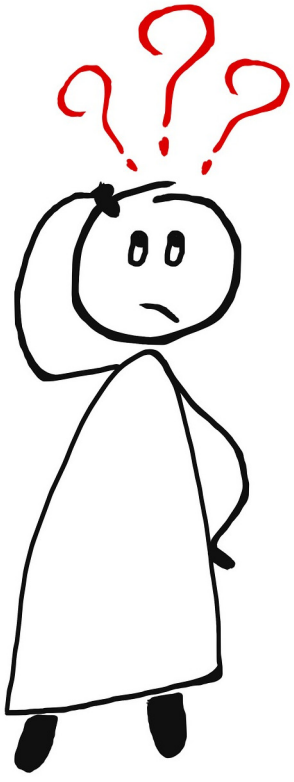
$$M = \langle Q_1 \times Q_2, \Sigma_1 \cap \Sigma_2, \delta, \langle q_{01}, q_{02}\rangle, E_1 \times E_2\rangle$$

$$\text{with } \delta(\langle q_1, q_2\rangle, a) = \langle \delta_1(q_1, a), \delta_2(q_2, a)\rangle$$

accepts $\mathcal{L}(M) = \mathcal{L}(M_1) \cap \mathcal{L}(M_2)$. □

German: Kreuzproduktautomat

Questions



Questions?

Decidability

Decision Problems and Decidability (1)

"Intuitive Definition:" Decision Problem, Decidability

A decision problem is an algorithmic problem where

- for a given input
- an algorithm determines if the input has a given property
- and then produces the output "yes" or "no" accordingly.

A decision problem is decidable if an algorithm for it (that always gives the correct answer) exists.

German: Entscheidungsproblem, Eingabe, Eigenschaft, Ausgabe, entscheidbar

Note: "exists" $\neq$ "is known"

Decision Problems and Decidability (2)

Notes:

- not a formal definition: we did not formally define “algorithm”, “input”, “output” etc. (which is not trivial)
- lack of a formal definition makes it difficult to prove that something is not decidable

$\rightsquigarrow$ studied thoroughly in the next part of the course

Decision Problems: Example

For now we describe decision problems in a semi-formal "given" / "question" way:

Example (Emptiness Problem for Regular Languages)

The emptiness problem $\mathrm{P}_\emptyset$ for regular languages is the following problem:

Given:	regular grammar G
Question:	Is $\mathcal{L}(G) = \emptyset$?

German: Leerheitsproblem

Word Problem

Definition (Word Problem for Regular Languages)

The word problem $\mathrm{P}_{\in}$ for regular languages is:

Given:	regular grammar G with alphabet Σ and word $w \in \Sigma^*$
Question:	Is $w \in \mathcal{L}(G)$?

German: Wortproblem (für reguläre Sprachen)

Decidability: Word Problem

Theorem

The word problem for regular languages is decidable.

Proof.

Construct a DFA M with $\mathcal{L}(M) = \mathcal{L}(G)$.
(The proofs in Chapter C2 describe a possible method.)

Simulate M on input w. The simulation ends after $|w|$ steps.

The DFA M is an end state after this iff $w \in \mathcal{L}(G)$.
Print "yes" or "no" accordingly. □

Emptiness Problem

Definition (Emptiness Problem for Regular Languages)

The emptiness problem $\mathrm{P}_\emptyset$ for regular languages is:

Given: regular grammar G
Question: Is $\mathcal{L}(G) = \emptyset$?

German: Leerheitsproblem

Decidability: Emptiness Problem

Theorem

The emptiness problem for regular languages is decidable.

Proof.

Construct a DFA M with $\mathcal{L}(M) = \mathcal{L}(G)$.

We have $\mathcal{L}(G) = \emptyset$ iff in the transition diagram of M there is no path from the start state to any end state.

This can be checked with standard graph algorithms (e.g., breadth-first search). □

Finiteness Problem

Definition (Finiteness Problem for Regular Languages)

The finiteness problem P_{∞} for regular languages is:

Given: regular grammar G
Question: Is $|\mathcal{L}(G)| < \infty$?

German: Endlichkeitsproblem

Decidability: Finiteness Problem

Theorem

The finiteness problem for regular languages is decidable.

Proof.

Construct a DFA M with $\mathcal{L}(M) = \mathcal{L}(G)$.

We have $|\mathcal{L}(G)| = \infty$ iff in the transition diagram of M there is a cycle that is reachable from the start state and from which an end state can be reached.

This can be checked with standard graph algorithms. □

Intersection Problem

Definition (Intersection Problem for Regular Languages)

The intersection problem $\mathrm{P}_\cap$ for regular languages is:

Given: regular grammars G and G'

Question: Is $\mathcal{L}(G) \cap \mathcal{L}(G') = \emptyset$?

German: Schnittproblem

Decidability: Intersection Problem

Theorem

The intersection problem for regular languages is decidable.

Proof.

Using the closure of regular languages under intersection,
we can construct (e.g., by converting to DFAs, constructing the product automaton, then converting back to a grammar)
a grammar G'' with $\mathcal{L}(G'') = \mathcal{L}(G) \cap \mathcal{L}(G')$
and use the algorithm for the emptiness problem $\mathrm{P}_\emptyset$. □

Equivalence Problem

Definition (Equivalence Problem for Regular Languages)

The equivalence problem $\mathrm{P}_=$ for regular languages is:

Given: regular grammars G and G'
Question: Is $\mathcal{L}(G) = \mathcal{L}(G')$?

German: Äquivalenzproblem

Decidability: Equivalence Problem

Theorem

The equivalence problem for regular languages is decidable.

Proof.

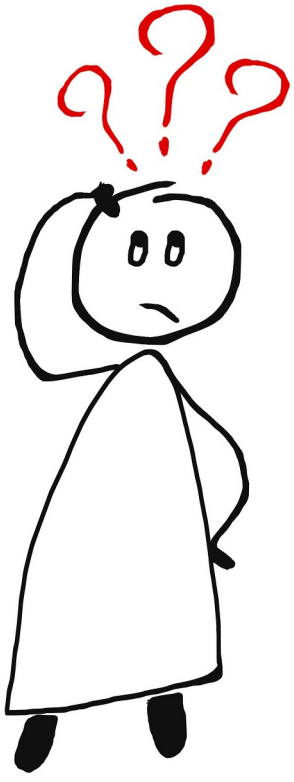
In general for languages L and L', we have

$$L = L' \text{ iff } (L \cap \bar{L}') \cup (\bar{L} \cap L') = \emptyset.$$

The regular languages are closed under intersection, union and complement, and we know algorithms for these operations.

We can therefore construct a grammar for $(L \cap \bar{L}') \cup (\bar{L} \cap L')$ and use the algorithm for the emptiness problem $\mathrm{P}_\emptyset$. □

Questions



Questions?

Summary

Summary

- Minimal automata are the smallest possible DFAs for a given language and are unique for each language.
- The regular languages are closed under all usual operations (union, intersection, complement, product, star).
- All usual decision problems (word problem, emptiness, finiteness, intersection, equivalence) are decidable for regular languages.