# Theory of Computer Science

## C4. Regular Languages: Minimal Automata, Closure Properties and Decidability

Gabriele Röger

University of Basel

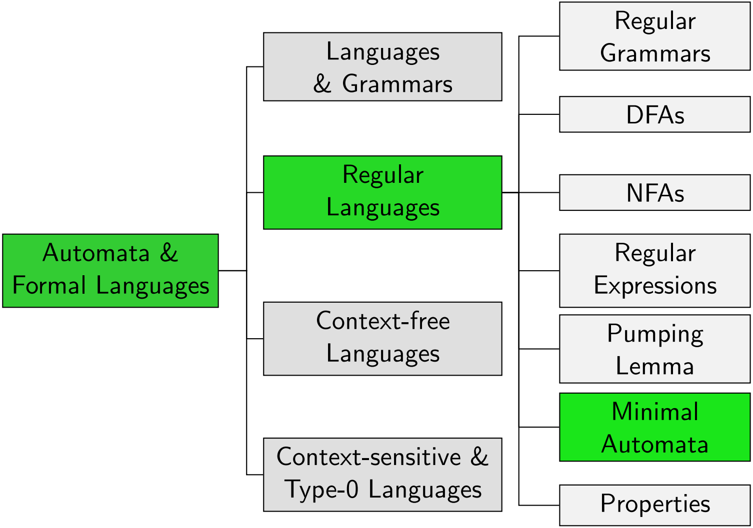
March 27, 2019

## C4.1 Minimal Automata

## C4.2 Closure Properties

## C4.3 Decidability

# Overview



Automata & Formal Languages

Languages & Grammars

Regular Languages

Context-free Languages

Context-sensitive & Type-0 Languages

Regular Grammars

DFAs

NFAs

Regular Expressions

Pumping Lemma

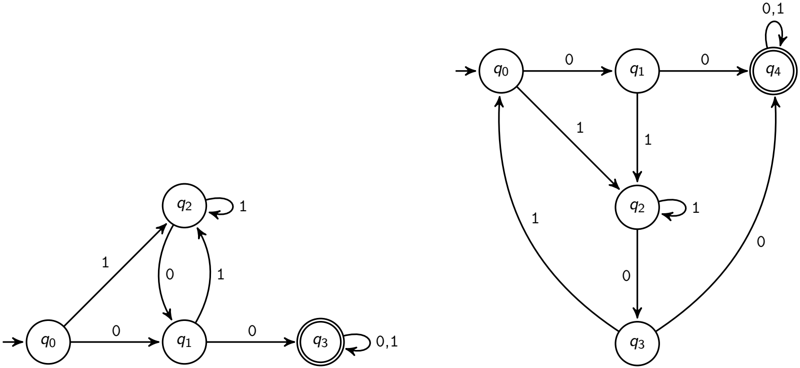
Minimal Automata

Properties

# C4.1 Minimal Automata

# Example

The following DFAs accept the same language:



Question: What is the smallest DFA that accepts this language?

# Minimal Automaton: Definition

**Definition**

A minimal automaton for a regular language $L$ is a DFA $M = \langle Q, \Sigma, \delta, q_0, E \rangle$ with $\mathcal{L}(M) = L$ and a minimal number of states.

This means there is no DFA $M' = \langle Q', \Sigma, \delta', q'_0, E' \rangle$ with $\mathcal{L}(M) = \mathcal{L}(M')$ and $|Q'| < |Q|$.

How to find a minimal automaton?

Idea:

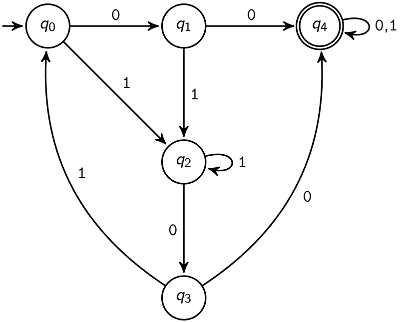
- Start with any DFA that accepts the language.
- Merge states from which exactly the same words lead to an end state.

# Minimal Automaton: Algorithm

Input: DFA $M$
(without states that are unreachable from the start state)
Output: list of states that have to be merged
to obtain an equivalent minimal automaton

1. Create table of all pairs of states $\{q, q'\}$ with $q \neq q'$.
2. Mark all pairs $\{q, q'\}$ with $q \in E$ and $q' \notin E$.
3. If there is an unmarked pair $\{q, q'\}$ where $\{\delta(q, a), \delta(q', a)\}$ for some $a \in \Sigma$ is already marked, then also mark $\{q, q'\}$.
4. Repeat the last step until there are no more changes.
5. All states in pairs that are still unmarked can be merged into one state.

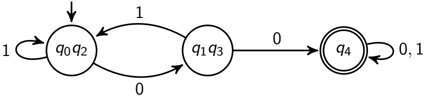
# Minimal Automaton: Example



| | $q_0$ | $q_1$ | $q_2$ | $q_3$ |
|---|---|---|---|---|
| $q_1$ | × | | | |
| $q_2$ | | × | | |
| $q_3$ | × | | × | |
| $q_4$ | × | × | × | × |

States $q_0, q_2$ and $q_1, q_3$ can be merged into one state each.

Result:



# Computation and Uniqueness of Minimal Automata

**Theorem**

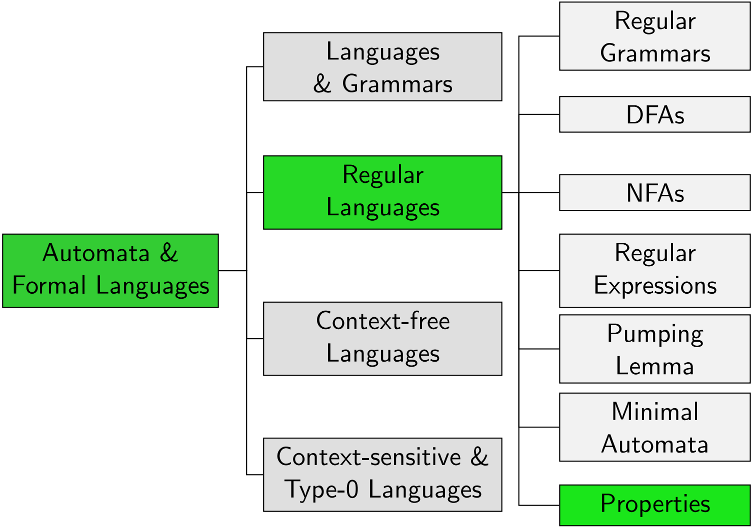
*The algorithm described on the previous slides produces a minimal automaton for the language accepted by the given input DFA.*

**Theorem**

*All minimal automata for a language $L$ are unique
up to isomorphism (i.e., renaming of states).*

Without proof.

# Overview



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# C4.2 Closure Properties

# Closure Properties

How can you combine
regular languages in a way to get
another regular language
as a result?

Picture courtesy of stockimages / FreeDigitalPhotos.net

# Closure Properties: Operations

Let $L$ and $L'$ be regular languages over $\Sigma$ and $\Sigma'$, respectively.

We consider the following operations:

- union $L \cup L' = \{w \mid w \in L \text{ or } w \in L'\}$ over $\Sigma \cup \Sigma'$
- intersection $L \cap L' = \{w \mid w \in L \text{ and } w \in L'\}$ over $\Sigma \cap \Sigma'$
- complement $\bar{L} = \{w \in \Sigma^* \mid w \notin L\}$ over $\Sigma$
- product $LL' = \{uv \mid u \in L \text{ and } v \in L'\}$ over $\Sigma \cup \Sigma'$
  - special case: $L^n = L^{n-1}L$, where $L^0 = \{\varepsilon\}$
- star $L^* = \bigcup_{k \geq 0} L^k$ over $\Sigma$

German: Abschlusseigenschaften, Vereinigung, Schnitt, Komplement, Produkt, Stern

# Closure Properties

## Definition (Closure)

Let $\mathcal{K}$ be a class of languages.

Then $\mathcal{K}$ is closed...

- ...under union if $L, L' \in \mathcal{K}$ implies $L \cup L' \in \mathcal{K}$
- ...under intersection if $L, L' \in \mathcal{K}$ implies $L \cap L' \in \mathcal{K}$
- ...under complement if $L \in \mathcal{K}$ implies $\bar{L} \in \mathcal{K}$
- ...under product if $L, L' \in \mathcal{K}$ implies $LL' \in \mathcal{K}$
- ...under star if $L \in \mathcal{K}$ implies $L^* \in \mathcal{K}$

German: Abgeschlossenheit, $\mathcal{K}$ ist abgeschlossen unter Vereinigung (Schnitt, Komplement, Produkt, Stern)

# Clourse Properties of Regular Languages

**Theorem**

*The regular languages are closed under:*

- *union*
- *intersection*
- *complement*
- *product*
- *star*

# Closure Properties

Proof.

Closure under union, product, and star follows because for regular expressions $\alpha$ and $\beta$, the expressions $(\alpha|\beta)$, $(\alpha\beta)$ and $(\alpha^*)$ describe the corresponding languages.

Complement: Let $M = \langle Q, \Sigma, \delta, q_0, E\rangle$ be a DFA with $\mathcal{L}(M) = L$. Then $M' = \langle Q, \Sigma, \delta, q_0, Q \setminus E\rangle$ is a DFA with $\mathcal{L}(M') = \bar{L}$.

Intersection: Let $M_1 = \langle Q_1, \Sigma_1, \delta_1, q_{01}, E_1\rangle$ and $M_2 = \langle Q_2, \Sigma_2, \delta_2, q_{02}, E_2\rangle$ be DFAs. The product automaton

$$M = \langle Q_1 \times Q_2, \Sigma_1 \cap \Sigma_2, \delta, \langle q_{01}, q_{02}\rangle, E_1 \times E_2\rangle$$

$$\text{with } \delta(\langle q_1, q_2\rangle, a) = \langle \delta_1(q_1, a), \delta_2(q_2, a)\rangle$$

accepts $\mathcal{L}(M) = \mathcal{L}(M_1) \cap \mathcal{L}(M_2)$. ☐

German: Kreuzproduktautomat

# C4.3 Decidability

# Decision Problems and Decidability (1)

### "Intuitive Definition:" Decision Problem, Decidability

A decision problem is an algorithmic problem where

- for a given input
- an algorithm determines if the input has a given property
- and then produces the output "yes" or "no" accordingly.

A decision problem is decidable if an algorithm for it (that always gives the correct answer) exists.

German: Entscheidungsproblem, Eingabe, Eigenschaft, Ausgabe, entscheidbar

Note: "exists" $\neq$ "is known"

# Decision Problems and Decidability (2)

Notes:

- not a formal definition: we did not formally define “algorithm”, “input”, “output” etc. (which is not trivial)
- lack of a formal definition makes it difficult to prove that something is not decidable

⇝ studied thoroughly in the next part of the course

# Decision Problems: Example

For now we describe decision problems in a semi-formal "given" / "question" way:

**Example (Emptiness Problem for Regular Languages)**

The emptiness problem $\mathrm{P}_\emptyset$ for regular languages is the following problem:

Given: regular grammar $G$
Question: Is $\mathcal{L}(G) = \emptyset$?

German: Leerheitsproblem

# Word Problem

**Definition (Word Problem for Regular Languages)**

The word problem $\mathrm{P}_{\in}$ for regular languages is:

| | |
|---|---|
| Given: | regular grammar $G$ with alphabet $\Sigma$<br>and word $w \in \Sigma^*$ |
| Question: | Is $w \in \mathcal{L}(G)$? |

German: Wortproblem (für reguläre Sprachen)

# Decidability: Word Problem

**Theorem**

*The word problem for regular languages is decidable.*

**Proof.**

Construct a DFA $M$ with $\mathcal{L}(M) = \mathcal{L}(G)$.
(The proofs in Chapter C2 describe a possible method.)

Simulate $M$ on input $w$. The simulation ends after $|w|$ steps.

The DFA $M$ is an end state after this iff $w \in \mathcal{L}(G)$.
Print "yes" or "no" accordingly. □

# Emptiness Problem

**Definition (Emptiness Problem for Regular Languages)**

The emptiness problem $\mathrm{P}_\emptyset$ for regular languages is:

Given: regular grammar $G$
Question: Is $\mathcal{L}(G) = \emptyset$?

German: Leerheitsproblem

# Decidability: Emptiness Problem

**Theorem**

*The emptiness problem for regular languages is decidable.*

**Proof.**

Construct a DFA $M$ with $\mathcal{L}(M) = \mathcal{L}(G)$.

We have $\mathcal{L}(G) = \emptyset$ iff in the transition diagram of $M$ there is no path from the start state to any end state.

This can be checked with standard graph algorithms (e.g., breadth-first search). □

# Finiteness Problem

**Definition (Finiteness Problem for Regular Languages)**

The finiteness problem $\mathrm{P}_{\not\infty}$ for regular languages is:

| | |
|---|---|
| Given: | regular grammar $G$ |
| Question: | Is $\lvert\mathcal{L}(G)\rvert < \infty$? |

German: Endlichkeitsproblem

# Decidability: Finiteness Problem

**Theorem**

*The finiteness problem for regular languages is decidable.*

**Proof.**

Construct a DFA $M$ with $\mathcal{L}(M) = \mathcal{L}(G)$.

We have $|\mathcal{L}(G)| = \infty$ iff in the transition diagram of $M$ there is a cycle that is reachable from the start state and from which an end state can be reached.

This can be checked with standard graph algorithms. $\square$

# Intersection Problem

**Definition (Intersection Problem for Regular Languages)**

The intersection problem $\mathrm{P}_{\cap}$ for regular languages is:

Given: regular grammars $G$ and $G'$
Question: Is $\mathcal{L}(G) \cap \mathcal{L}(G') = \emptyset$?

German: Schnittproblem

# Decidability: Intersection Problem

**Theorem**

*The intersection problem for regular languages is decidable.*

**Proof.**

Using the closure of regular languages under intersection, we can construct (e.g., by converting to DFAs, constructing the product automaton, then converting back to a grammar) a grammar $G''$ with $\mathcal{L}(G'') = \mathcal{L}(G) \cap \mathcal{L}(G')$ and use the algorithm for the emptiness problem $\mathrm{P}_{\emptyset}$. $\square$

# Equivalence Problem

Definition (Equivalence Problem for Regular Languages)

The equivalence problem $\mathrm{P}_{=}$ for regular languages is:

| | |
|---|---|
| Given: | regular grammars $G$ and $G'$ |
| Question: | Is $\mathcal{L}(G) = \mathcal{L}(G')$? |

German: Äquivalenzproblem

# Decidability: Equivalence Problem

**Theorem**

*The equivalence problem for regular languages is decidable.*

**Proof.**

In general for languages $L$ and $L'$, we have

$$L = L' \text{ iff } (L \cap \bar{L}') \cup (\bar{L} \cap L') = \emptyset.$$

The regular languages are closed under intersection, union and complement, and we know algorithms for these operations.

We can therefore construct a grammar for $(L \cap \bar{L}') \cup (\bar{L} \cap L')$ and use the algorithm for the emptiness problem $\mathrm{P}_\emptyset$. $\square$

# Summary

- Minimal automata are the smallest possible DFAs for a given language and are unique for each language.
- The regular languages are closed under all usual operations (union, intersection, complement, product, star).
- All usual decision problems (word problem, emptiness, finiteness, intersection, equivalence) are decidable for regular languages.