Theory of Computer Science

B5. Predicate Logic II

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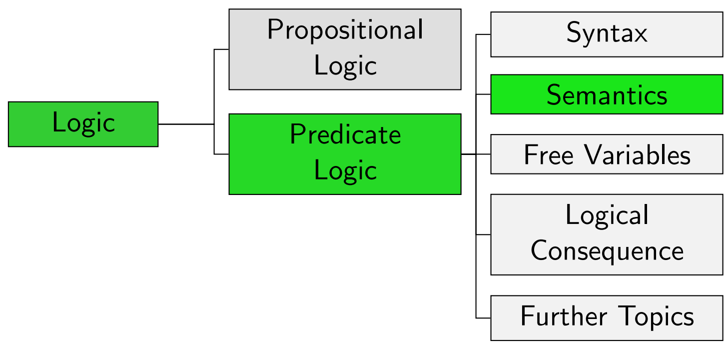
B5.1 Semantics of Predicate Logic

Logic: Overview



Semantics: Motivation

- interpretations in propositional logic: truth assignments for the propositional variables
- There are no propositional variables in predicate logic.
- instead: interpretation determines meaning of the constant, function and predicate symbols.
- meaning of variable symbols not determined by interpretation but by separate variable assignment.

Interpretations and Variable Assignments

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation, Variable Assignment)

An interpretation (for $\mathcal{S}$) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:

- a non-empty set U called the universe and
- a function $\cdot^{\mathcal{I}}$ that assigns a meaning to the constant, function, and predicate symbols:
  - $\mathsf{c}^{\mathcal{I}} \in U$ for constant symbols $\mathsf{c} \in \mathcal{C}$
  - $\mathsf{f}^{\mathcal{I}} : U^k \to U$ for k-ary function symbols $\mathsf{f} \in \mathcal{F}$
  - $\mathsf{P}^{\mathcal{I}} \subseteq U^k$ for k-ary predicate symbols $\mathsf{P} \in \mathcal{P}$

A variable assignment (for $\mathcal{S}$ and universe U) is a function $\alpha : \mathcal{V} \to U$.

German: Interpretation, Variablenzuweisung, Universum (or Grundmenge)

Interpretations and Variable Assignments: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$,
$\mathcal{C} = \{\mathsf{zero}, \mathsf{one}\}$, $\mathcal{F} = \{\mathsf{sum}, \mathsf{product}\}$, $\mathcal{P} = \{\mathsf{SquareNumber}\}$
$ar(\mathsf{sum}) = ar(\mathsf{product}) = 2$, $ar(\mathsf{SquareNumber}) = 1$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\mathsf{zero}^{\mathcal{I}} = u_0$
- $\mathsf{one}^{\mathcal{I}} = u_1$
- $\mathsf{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\mathsf{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\mathsf{SquareNumber}^{\mathcal{I}} = \{u_0, u_1, u_2, u_4\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Semantics: Informally

Example: $(\forall x(\mathsf{Block}(x) \to \mathsf{Red}(x)) \land \mathsf{Block}(\mathsf{a}))$
"For all objects x: if x is a block, then x is red.
Also, the object called a is a block."

- Terms are interpreted as objects.
- Unary predicates denote properties of objects (to be a block, to be red, to be a square number, ...)
- General predicates denote relations between objects (to be someone's child, to have a common divisor, ...)
- Universally quantified formulas ("∀") are true if they hold for every object in the universe.
- Existentially quantified formulas ("∃") are true if they hold for at least one object in the universe.

Interpretations of Terms

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation of a Term)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.

Let t be a term over $\mathcal{S}$.
The interpretation of t under $\mathcal{I}$ and α, written as $t^{\mathcal{I},\alpha}$,
is the element of the universe U defined as follows:

- If $t = x$ with $x \in \mathcal{V}$ (t is a variable term):
  $x^{\mathcal{I},\alpha} = \alpha(x)$
- If $t = \mathsf{c}$ with $\mathsf{c} \in \mathcal{C}$ (t is a constant term):
  $\mathsf{c}^{\mathcal{I},\alpha} = \mathsf{c}^{\mathcal{I}}$
- If $t = \mathsf{f}(t_1, \dots, t_k)$ (t is a function term):
  $\mathsf{f}(t_1, \dots, t_k)^{\mathcal{I},\alpha} = \mathsf{f}^{\mathcal{I}}(t_1^{\mathcal{I},\alpha}, \dots, t_k^{\mathcal{I},\alpha})$

Interpretations of Terms: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\mathsf{zero}, \mathsf{one}\}$, $\mathcal{F} = \{\mathsf{sum}, \mathsf{product}\}$,
$ar(\mathsf{sum}) = ar(\mathsf{product}) = 2$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\mathsf{zero}^{\mathcal{I}} = u_0$
- $\mathsf{one}^{\mathcal{I}} = u_1$
- $\mathsf{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- $\mathsf{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Interpretations of Terms: Example (ctd.)

Example (ctd.)

- $\mathsf{zero}^{\mathcal{I},\alpha} =$
- $y^{\mathcal{I},\alpha} =$
- $\mathsf{sum}(x, y)^{\mathcal{I},\alpha} =$
- $\mathsf{product}(\mathsf{one}, \mathsf{sum}(x, \mathsf{zero}))^{\mathcal{I},\alpha} =$

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.
We say that $\mathcal{I}$ and α satisfy a predicate logic formula φ
(also: φ is true under $\mathcal{I}$ and α), written: $\mathcal{I}, \alpha \models \varphi$,
according to the following inductive rules:

$$
\begin{aligned}
\mathcal{I}, \alpha \models \mathsf{P}(t_1, \dots, t_k) \quad & \text{iff } \langle t_1^{\mathcal{I},\alpha}, \dots, t_k^{\mathcal{I},\alpha} \rangle \in \mathsf{P}^{\mathcal{I}} \\
\mathcal{I}, \alpha \models (t_1 = t_2) \quad & \text{iff } t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha} \\
\mathcal{I}, \alpha \models \neg\varphi \quad & \text{iff } \mathcal{I}, \alpha \not\models \varphi \\
\mathcal{I}, \alpha \models (\varphi \land \psi) \quad & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models (\varphi \lor \psi) \quad & \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \qquad \dots
\end{aligned}
$$

German: $\mathcal{I}$ und α erfüllen φ (also: φ ist wahr unter $\mathcal{I}$ und α)

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

...

$$\mathcal{I}, \alpha \models \forall x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U$$
$$\mathcal{I}, \alpha \models \exists x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U$$

where $\alpha[x := u]$ is the same variable assignment as α, except that it maps variable x to the value u.
Formally:

$$(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$$

Semantics: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\mathrm{a}, \mathrm{b}\}$, $\mathcal{F} = \emptyset$, $\mathcal{P} = \{\mathrm{Block}, \mathrm{Red}\}$, $ar(\mathrm{Block}) = ar(\mathrm{Red}) = 1$.

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_1, u_2, u_3, u_4, u_5\}$
- $\mathrm{a}^{\mathcal{I}} = u_1$
- $\mathrm{b}^{\mathcal{I}} = u_3$
- $\mathrm{Block}^{\mathcal{I}} = \{u_1, u_2\}$
- $\mathrm{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$

$\alpha = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_1\}$

Semantics: Example (ctd.)

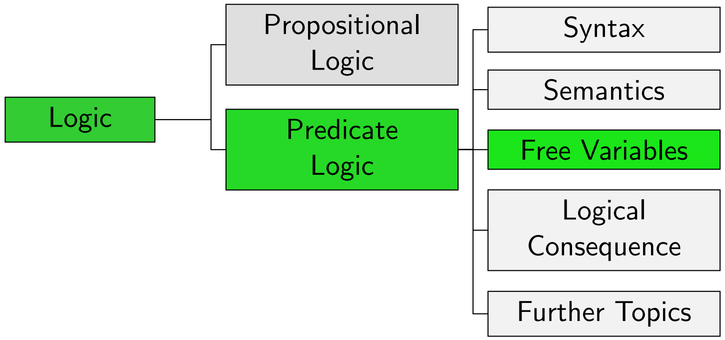
Example (ctd.)

Questions:

- $\mathcal{I}, \alpha \models (\mathrm{Block}(\mathrm{b}) \lor \neg \mathrm{Block}(\mathrm{b}))$?
- $\mathcal{I}, \alpha \models (\mathrm{Block}(x) \rightarrow (\mathrm{Block}(x) \lor \neg \mathrm{Block}(y)))$?
- $\mathcal{I}, \alpha \models (\mathrm{Block}(\mathrm{a}) \land \mathrm{Block}(\mathrm{b}))$?
- $\mathcal{I}, \alpha \models \forall x (\mathrm{Block}(x) \rightarrow \mathrm{Red}(x))$?

B5.2 Free and Bound Variables

Logic: Overview



Free and Bound Variables: Motivation

Question:

- Consider a signature with variable symbols $\{x_1, x_2, x_3, \dots\}$ and an interpretation $\mathcal{I}$.
- Which parts of the definition of α are relevant to decide whether $\mathcal{I}, \alpha \models (\forall x_4(\mathsf{R}(x_4, x_2) \lor (\mathsf{f}(x_3) = x_4)) \lor \exists x_3 \mathsf{S}(x_3, x_2))$?
- $\alpha(x_1)$, $\alpha(x_5)$, $\alpha(x_6)$, $\alpha(x_7)$, … are irrelevant since those variable symbols occur in no formula.
- $\alpha(x_4)$ also is irrelevant: the variable occurs in the formula, but all occurrences are bound by a surrounding quantifier.
- $\rightsquigarrow$ only assignments for free variables x_2 and x_3 relevant

German: gebundene und freie Variablen

Variables of a Term

Definition (Variables of a Term)

Let t be a term. The set of variables that occur in t, written as $var(t)$, is defined as follows:

- $var(x) = \{x\}$ for variable symbols x
- $var(\mathsf{c}) = \emptyset$ for constant symbols c
- $var(\mathsf{f}(t_1, \dots, t_l)) = var(t_1) \cup \dots \cup var(t_l)$ for function terms

terminology: A term t with $var(t) = \emptyset$ is called ground term.
German: Grundterm

example: $var(\mathsf{product}(x, \mathsf{sum}(\mathsf{k}, y))) =$

Free and Bound Variables of a Formula

Definition (Free Variables)

Let φ be a predicate logic formula. The set of free variables of φ, written as $free(\varphi)$, is defined as follows:

- $free(\mathsf{P}(t_1, \dots, t_k)) = var(t_1) \cup \dots \cup var(t_k)$
- $free((t_1 = t_2)) = var(t_1) \cup var(t_2)$
- $free(\neg\varphi) = free(\varphi)$
- $free((\varphi \land \psi)) = free((\varphi \lor \psi)) = free(\varphi) \cup free(\psi)$
- $free(\forall x\, \varphi) = free(\exists x\, \varphi) = free(\varphi) \setminus \{x\}$

Example: $free((\forall x_4(\mathsf{R}(x_4, x_2) \lor (\mathsf{f}(x_3) = x_4)) \lor \exists x_3 \mathsf{S}(x_3, x_2)))$
$=$

Closed Formulas/Sentences

Note: Let φ be a formula and let α and β variable assignments with $\alpha(x) = \beta(x)$ for all free variables x of φ.

Then $\mathcal{I}, \alpha \models \varphi$ iff $\mathcal{I}, \beta \models \varphi$.

In particular, α is completely irrelevant if $free(\varphi) = \emptyset$.

Definition (Closed Formulas/Sentences)

A formula φ without free variables (i. e., $free(\varphi) = \emptyset$) is called closed formula or sentence.

If φ is a sentence, then we often write $\mathcal{I} \models \varphi$ instead of $\mathcal{I}, \alpha \models \varphi$, since the definition of α does not influence whether φ is true under $\mathcal{I}$ and α or not.

Formulas with at least one free variable are called open.

German: geschlossene Formel/Satz, offene Formel

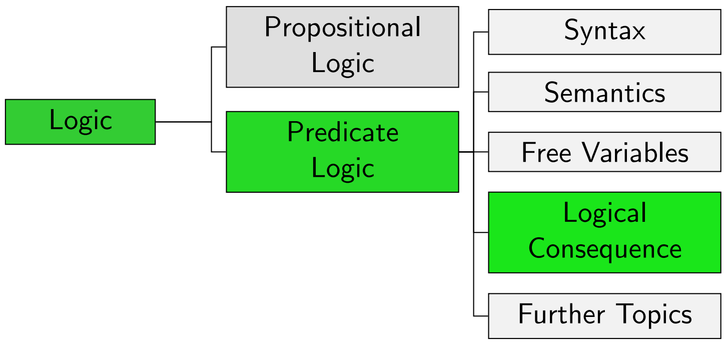
Closed Formulas/Sentences: Examples

Question: Which of the following formulas are sentences?

- $(\text{Block}(b) \lor \neg\text{Block}(b))$
- $(\text{Block}(x) \to (\text{Block}(x) \lor \neg\text{Block}(y)))$
- $(\text{Block}(a) \land \text{Block}(b))$
- $\forall x(\text{Block}(x) \to \text{Red}(x))$

B5.3 Logical Consequences

Logic: Overview



Terminology for Formulas

The terminology we introduced for propositional logic similarly applies to predicate logic:

- Interpretation $\mathcal{I}$ and variable assignment α form a model of the formula φ if $\mathcal{I}, \alpha \models \varphi$.
- Formula φ is satisfiable if $\mathcal{I}, \alpha \models \varphi$ for at least one $\mathcal{I}, \alpha$.
- Formula φ is falsifiable if $\mathcal{I}, \alpha \not\models \varphi$. for at least one $\mathcal{I}, \alpha$
- Formula φ is valid if $\mathcal{I}, \alpha \models \varphi$ for all $\mathcal{I}, \alpha$.
- Formula φ is unsatisfiable if $\mathcal{I}, \alpha \not\models \varphi$ for all $\mathcal{I}, \alpha$.
- Formulas φ and ψ are logically equivalent, written as $\varphi \equiv \psi$, if they have the same models.

German: Modell, erfüllbar, falsifizierbar, gültig, unerfüllbar, logisch äquivalent

Sets of Formulas: Semantics

Definition (Satisfied/True Sets of Formulas)

Let $\mathcal{S}$ be a signature, Φ a set of formulas over $\mathcal{S}$, $\mathcal{I}$ an interpretation for $\mathcal{S}$ and α a variable assignment for $\mathcal{S}$ and the universe of $\mathcal{I}$.

We say that $\mathcal{I}$ and α satisfy the formulas Φ (also: Φ is true under $\mathcal{I}$ and α), written as: $\mathcal{I}, \alpha \models \Phi$, if $\mathcal{I}, \alpha \models \varphi$ for all $\varphi \in \Phi$.

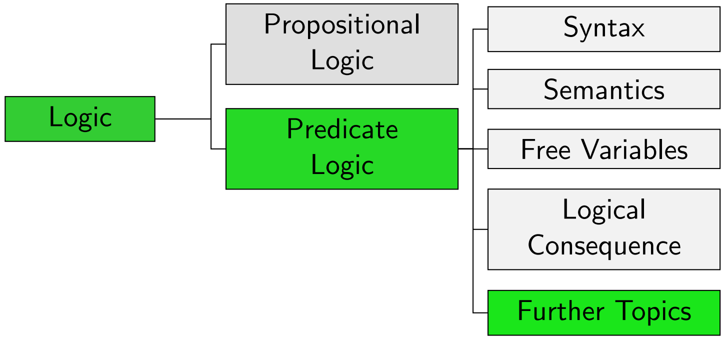
German: $\mathcal{I}$ und α erfüllen Φ, Φ ist wahr unter $\mathcal{I}$ und α

Terminology for Sets of Formulas and Sentences

- Again, we use the same notations and concepts as in propositional logic.
  Example:
  - A set of formulas Φ is satisfiable if $\mathcal{I}, \alpha \models \Phi$ for at least one $\mathcal{I}, \alpha$.
  - A set of formulas Φ (logically) implies formula ψ, written as $\Phi \models \psi$, if all models of Φ are models of ψ.
- All concepts can be used for the special case of sentences (or sets of sentences). In this case we usually omit α.
  Examples:
  - Interpretation $\mathcal{I}$ is a model of a sentence φ if $\mathcal{I} \models \varphi$.
  - Sentence φ is unsatisfiable if $\mathcal{I} \not\models \varphi$ for all $\mathcal{I}$.
- similarly:
  - $\varphi \models \psi$ if $\{\varphi\} \models \psi$
  - $\Phi \models \Psi$ if $\Phi \models \psi$ for all $\psi \in \Psi$

B5.4 Further Topics

Logic: Overview



Further Topics

Based on these definitions we could cover the same topics as in propositional logic:

- important logical equivalences
- normal forms
- theorems about reasoning (deduction theorem etc.)

We briefly discuss some general results on those topics but will not go into detail.

Logical Equivalences

- All logical equivalences of propositional logic also hold in predicate logic (e. g., $(\varphi \lor \psi) \equiv (\psi \lor \varphi)$). (Why?)
- Additionally the following equivalences and implications hold:

$$
\begin{aligned}
(\forall x\varphi \land \forall x\psi) &\equiv \forall x(\varphi \land \psi) && \\
(\forall x\varphi \lor \forall x\psi) &\models \forall x(\varphi \lor \psi) && \text{but not vice versa} \\
(\forall x\varphi \land \psi) &\equiv \forall x(\varphi \land \psi) && \text{if } x \notin \mathit{free}(\psi) \\
(\forall x\varphi \lor \psi) &\equiv \forall x(\varphi \lor \psi) && \text{if } x \notin \mathit{free}(\psi) \\
\neg\forall x\varphi &\equiv \exists x\neg\varphi && \\
\exists x(\varphi \lor \psi) &\equiv (\exists x\varphi \lor \exists x\psi) && \\
\exists x(\varphi \land \psi) &\models (\exists x\varphi \land \exists x\psi) && \text{but not vice versa} \\
(\exists x\varphi \lor \psi) &\equiv \exists x(\varphi \lor \psi) && \text{if } x \notin \mathit{free}(\psi) \\
(\exists x\varphi \land \psi) &\equiv \exists x(\varphi \land \psi) && \text{if } x \notin \mathit{free}(\psi) \\
\neg\exists x\varphi &\equiv \forall x\neg\varphi &&
\end{aligned}
$$

Normal Forms

Analogously to DNF and CNF for propositional logic there are several normal forms for predicate logic, such as

- negation normal form (NNF): negation symbols ($\neg$) are only allowed in front of atoms
- prenex normal form: quantifiers must form the outermost part of the formula
- Skolem normal form: prenex normal form without existential quantifiers

German: Negationsnormalform, Pränexnormalform, Skolemnormalform

Normal Forms (ctd.)

Efficient methods transform formula φ

- into an equivalent formula in negation normal form,
- into an equivalent formula in prenex normal form, or
- into an equisatisfiable formula in Skolem normal form.

German: erfüllbarkeitsäquivalent

B5.5 Summary

Summary

bound vs. free variables:

- bound vs. free variables: to decide if $\mathcal{I}, \alpha \models \varphi$, only free variables in α matter
- sentences (closed formulas): formulas without free variables

Once the basic definitions are in place, predicate logic can be developed in the same way as propositional logic:

- logical consequences
- logical equivalences
- normal forms
- deduction theorem etc.

Other Logics

- We considered first-order predicate logic.
- Second-order predicate logic allows quantifying over predicate symbols.
- There are intermediate steps, e. g. monadic second-order logic (all quantified predicates are unary).
- Modal logics have new operators $\Box$ and $\Diamond$.
  - classical meaning: $\Box\varphi$ for "φ is necessary",
    $\Diamond\varphi$ for "φ is possible".
  - temporal logic: $\Box\varphi$ for "φ is always true in the future",
    $\Diamond\varphi$ for "φ is true at some point in the future"
  - deontic logic: $\Box\varphi$ for "φ is obligatory",
    $\Diamond\varphi$ for "φ is permitted"
  - ...
- In fuzzy logic, formulas are not true or false but have values between 0 and 1.

What's Next?

contents of this course:

A. background ✓
   - ▷ mathematical foundations and proof techniques

B. logic ✓
   - ▷ How can knowledge be represented?
     How can reasoning be automated?

C. automata theory and formal languages
   - ▷ What is a computation?

D. Turing computability
   - ▷ What can be computed at all?

E. complexity theory
   - ▷ What can be computed efficiently?

F. more computability theory
   - ▷ Other models of computability