Theory of Computer Science

B4. Predicate Logic I

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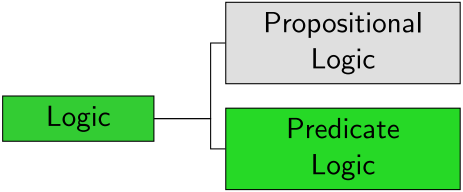
Logic: Overview



B4.1 Motivation

Limits of Propositional Logic

Cannot well be expressed in propositional logic:

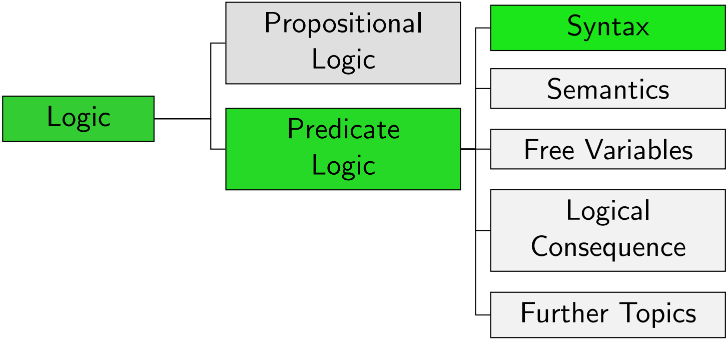
- "Everyone who does the exercises passes the exam."
- "If someone with administrator privileges presses 'delete', all data is gone."
- "Everyone has a mother."
- "If someone is the father of some person, the person is his child."

▷ need more expressive logic
$\rightsquigarrow$ predicate logic

German: Prädikatenlogik

B4.2 Syntax of Predicate Logic

Logic: Overview



Syntax: Building Blocks

- **Signatures** define allowed symbols.
  analogy: variable set A in propositional logic
- **Terms** are associated with objects by the semantics.
  no analogy in propositional logic
- **Formulas** are associated with truth values (true or false) by the semantics.
  analogy: formulas in propositional logic

German: Signatur, Term, Formel

Signatures: Definition

Definition (Signature)

A signature (of predicate logic) is a 4-tuple $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ consisting of the following four disjoint sets:

- a finite or countable set $\mathcal{V}$ of variable symbols
- a finite or countable set $\mathcal{C}$ of constant symbols
- a finite or countable set $\mathcal{F}$ of function symbols
- a finite or countable set $\mathcal{P}$ of predicate symbols (or relation symbols)

Every function symbol $\mathsf{f} \in \mathcal{F}$ and predicate symbol $\mathsf{P} \in \mathcal{P}$ has an associated arity $ar(\mathsf{f}), ar(\mathsf{P}) \in \mathbb{N}_0$ (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

Signatures: Terminology and Conventions

terminology:

- k-ary (function or predicate) symbol:
  symbol s with arity $ar(\mathrm{s}) = k$.
- also: unary, binary, ternary

German: k-stellig, unär, binär, ternär

conventions (in this lecture):

- variable symbols written in *italics*,
  other symbols upright.
- predicate symbols begin with capital letter,
  other symbols with lower-case letters

Signatures: Examples

Example: Arithmetic

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $\mathcal{C} = \{\mathsf{zero}, \mathsf{one}\}$
- $\mathcal{F} = \{\mathsf{sum}, \mathsf{product}\}$
- $\mathcal{P} = \{\mathsf{Positive}, \mathsf{SquareNumber}\}$

$ar(\mathsf{sum}) = ar(\mathsf{product}) = 2$, $ar(\mathsf{Positive}) = ar(\mathsf{SquareNumber}) = 1$

Signatures: Examples

Example: Genealogy

- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $\mathcal{C} = \{\text{roger-federer}, \text{lisa-simpson}\}$
- $\mathcal{F} = \emptyset$
- $\mathcal{P} = \{\text{Female}, \text{Male}, \text{Parent}\}$

$ar(\text{Female}) = ar(\text{Male}) = 1$, $ar(\text{Parent}) = 2$

Terms: Definition

Definition (Term)

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.
A term (over $\mathcal{S}$) is inductively constructed according to the following rules:

- Every variable symbol $v \in \mathcal{V}$ is a term.
- Every constant symbol $\mathsf{c} \in \mathcal{C}$ is a term.
- If $t_1, \dots, t_k$ are terms and $\mathsf{f} \in \mathcal{F}$ is a function symbol with arity k, then $\mathsf{f}(t_1, \dots, t_k)$ is a term.

German: Term

examples:

- x_4
- lisa-simpson
- $\mathsf{sum}(x_3, \mathsf{product}(\mathsf{one}, x_5))$

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over $\mathcal{S}$) is inductively defined as follows:

- If $t_1, \dots, t_k$ are terms (over $\mathcal{S}$) and $\mathsf{P} \in \mathcal{P}$ is a k-ary predicate symbol, then the atomic formula (or the atom) $\mathsf{P}(t_1, \dots, t_k)$ is a formula over $\mathcal{S}$.
- If t_1 and t_2 are terms (over $\mathcal{S}$), then the identity $(t_1 = t_2)$ is a formula over $\mathcal{S}$.
- If $x \in \mathcal{V}$ is a variable symbol and φ a formula over $\mathcal{S}$, then the universal quantification $\forall x\, \varphi$ and the existential quantification $\exists x\, \varphi$ are formulas over $\mathcal{S}$.

...

German: atomare Formel, Atom, Identität, Allquantifizierung, Existenzquantifizierung

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over $\mathcal{S}$) is inductively defined as follows:

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- If φ is a formula over $\mathcal{S}$, then so is its negation $\neg\varphi$.
- If φ and ψ are formulas over $\mathcal{S}$, then so are the conjunction $(\varphi \land \psi)$ and the disjunction $(\varphi \lor \psi)$.

German: Negation, Konjunktion, Disjunktion

Formulas: Examples

Examples: Arithmetic and Genealogy

- $\text{Positive}(x_2)$
- $\forall x\,(\neg\text{SquareNumber}(x) \lor \text{Positive}(x))$
- $\exists x_3\,(\text{SquareNumber}(x_3) \land \neg\text{Positive}(x_3))$
- $\forall x\,(x = y)$
- $\forall x\,(\text{sum}(x, x) = \text{product}(x, \text{one}))$
- $\forall x \exists y\,(\text{sum}(x, y) = \text{zero})$
- $\forall x \exists y\,(\text{Parent}(y, x) \land \text{Female}(y))$

Terminology: The symbols $\forall$ and $\exists$ are called quantifiers.

German: Quantoren

Abbreviations and Placement of Parentheses by Convention

abbreviations:

- $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \lor \psi)$.
- $(\varphi \leftrightarrow \psi)$ is an abbreviation for $((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi))$.
- Sequences of the same quantifier can be abbreviated. For example:
  - $\forall x \forall y \forall z\, \varphi \rightsquigarrow \forall xyz\, \varphi$
  - $\exists x \exists y \exists z\, \varphi \rightsquigarrow \exists xyz\, \varphi$
  - $\forall w \exists x \exists y \forall z\, \varphi \rightsquigarrow \forall w \exists xy \forall z\, \varphi$

placement of parentheses by convention:

- analogous to propositional logic
- quantifiers $\forall$ and $\exists$ bind more strongly than anything else.
- example: $\forall x\, \mathsf{P}(x) \rightarrow \mathsf{Q}(x)$ corresponds to $(\forall x\, \mathsf{P}(x) \rightarrow \mathsf{Q}(x))$,
  not $\forall x\, (\mathsf{P}(x) \rightarrow \mathsf{Q}(x))$.

B4.3 Summary

Summary

- Predicate logic is more expressive than propositional logic and allows statements over objects and their properties.
- Objects are described by terms that are built from variable, constant and function symbols.
- Properties and relations are described by formulas that are built from predicates, quantifiers and the usual logical operators.