Theory of Computer Science

B3. Propositional Logic III

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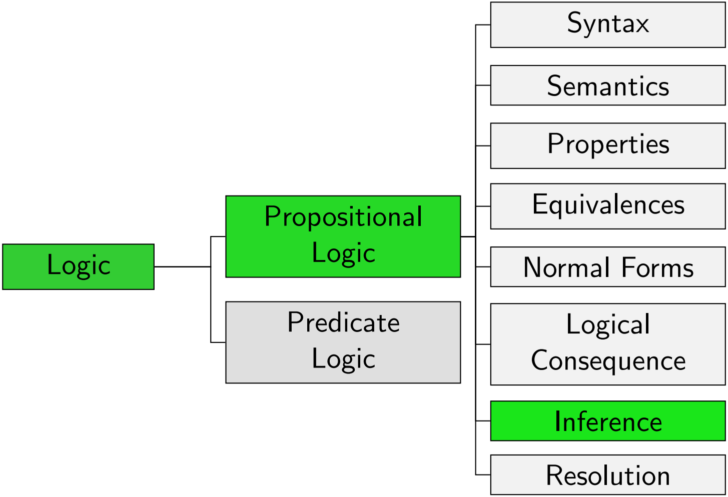
(Parts of) The Story So Far

- knowledge base: set of formulas describing given information; satisfiable, valid etc. used like for individual formulas
- logical consequence KB $\models \varphi$ means that φ is true whenever (= in all models where) KB is true

Inference

Logic: Overview



Inference: Motivation

- up to now: proof of logical consequence
  with semantic arguments

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- no general algorithm
- solution: produce with syntactic inference rules formulas that are logical consequences of given formulas.
- advantage: mechanical method can easily be implemented as an algorithm

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German: Inferenzregel

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German: Inferenzregel, Axiom

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- Meaning: "'Every model of $\varphi_1, \dots, \varphi_k$ is a model of ψ."'
- An axiom is an inference rule with $k = 0$.
- A set of syntactic inference rules is called a calculus or proof system.

German: Inferenzregel, Axiom, Kalkül, Beweissystem

Some Inference Rules for Propositional Logic

Modus ponens $$\frac{\varphi,\ (\varphi \rightarrow \psi)}{\psi}$$

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$\vee$-introduction $\dfrac{\varphi}{(\varphi \vee \psi)}$

$\leftrightarrow$-elimination $\dfrac{(\varphi \leftrightarrow \psi)}{(\varphi \rightarrow \psi)}$ $\dfrac{(\varphi \leftrightarrow \psi)}{(\psi \rightarrow \varphi)}$

Derivation

Definition (Derivation)

A derivation or proof of a formula φ from a knowledge base KB is a sequence of formulas $\psi_1, \dots, \psi_k$ with

- $\psi_k = \varphi$ and
- for all $i \in \{1, \dots, k\}$:
  - $\psi_i \in$ KB, or
  - ψ_i is the result of the application of an inference rule to elements from $\{\psi_1, \dots, \psi_{i-1}\}$.

German: Ableitung, Beweis

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \land R) \rightarrow S)\}$

Task: Find derivation of $(S \land R)$ from KB.

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Derivation: Example

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1. P (KB)
2. $(P \rightarrow Q)$ (KB)

Derivation: Example

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2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)

Derivation: Example

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Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

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2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$
Task: Find derivation of $(S \wedge R)$ from KB.

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2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
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Correctness and Completeness

Definition (Correctness and Completeness of a Calculus)

We write $\mathrm{KB} \vdash_C \varphi$ if there is a derivation of φ from KB in calculus C.
(If calculus C is clear from context, also only $\mathrm{KB} \vdash \varphi$.)

A calculus C is correct if for all KB and φ
$\mathrm{KB} \vdash_C \varphi$ implies $\mathrm{KB} \models \varphi$.

A calculus C is complete if for all KB and φ
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Consider calculus C, consisting of the derivation rules seen earlier.
Question: Is C correct?
Question: Is C complete?

German: korrekt, vollständig

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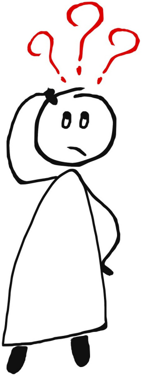
Definition (Refutation-Completeness)

A calculus C is refutation-complete if it holds for all unsatisfiable KB that $\mathrm{KB} \vdash_C \square$.

German: widerlegungsvollständig

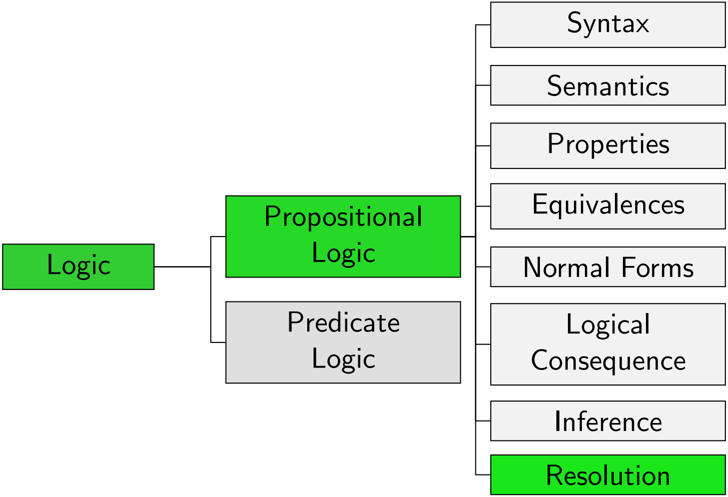
Questions



Questions?

Resolution Calculus

Logic: Overview



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- Show $\mathrm{KB} \models \varphi$ by derivability of $\mathrm{KB} \cup \{\neg\varphi\} \vdash_R \square$ with resolution calculus R.
- Resolution can require exponential time.
- This is probably the case for all refutation-complete proof methods. ⇝ complexity theory

German: Resolution, erfüllbarkeitsäquivalent

Knowledge Base as Set of Clauses

Simplified notation of knowledge bases in CNF

- Formula in CNF as set of clauses (due to commutativity, idempotence, associativity of $\wedge$)
- Set of formulas as set of clauses
- Clause as set of literals (due to commutativity, idempotence, associativity of $\vee$)
- Knowledge base as set of sets of literals

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Example

$$\mathrm{KB} = \{(P \vee P), ((\neg P \vee Q) \wedge (\neg P \vee R) \wedge (\neg P \vee Q) \wedge R), ((\neg Q \vee \neg R \vee S) \wedge P)\}$$

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as set of clauses:

$$\Delta = \{\{P\}, \{\neg P, Q\}, \{\neg P, R\}, \{R\}, \{\neg Q, \neg R, S\}\}$$

Resolution Rule

The resolution calculus consists of a single rule, called resolution rule:

$$\frac{C_1 \cup \{L\},\ C_2 \cup \{\neg L\}}{C_1 \cup C_2},$$

where C_1 und C_2 are (possibly empty) clauses and L is an atomic proposition.

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Terminology:

- L and $\neg L$ are the resolution literals,
- $C_1 \cup \{L\}$ and $C_2 \cup \{\neg L\}$ are the parent clauses, and
- $C_1 \cup C_2$ is the resolvent.

German: Resolutionskalkül, Resolutionsregel, Resolutionsliterale, Elternklauseln, Resolvent

Proof by Resolution

Definition (Proof by Resolution)

A proof by resolution of a clause D from a knowledge base Δ is a sequence of clauses $C_1, \dots, C_n$ with

- $C_n = D$ and
- for all $i \in \{1, \dots, n\}$:
  - $C_i \in \Delta$, or
  - C_i is resolvent of two clauses from $\{C_1, \dots, C_{i-1}\}$.

If there is a proof of D by resolution from Δ, we say that D can be derived with resolution from Δ and write $\Delta \vdash_R D$.

Remark: Resolution is a correct, refutation-complete, but incomplete calculus.

German: Resolutionsbeweis, “mit Resolution aus Δ abgeleitet”

Proof by Resolution: Example

Proof by Resolution for Testing a Logical Consequence: Example

Given: $\text{KB} = \{P, (P \rightarrow (Q \wedge R))\}$.

Show with resolution that $\text{KB} \models (R \vee S)$.

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Three steps:

1. Reduce logical consequence to unsatisfiability.
2. Transform knowledge base into clause form (CNF).
3. Derive empty clause $\square$ with resolution.

Proof by Resolution: Example

Proof by Resolution for Testing a Logical Consequence: Example

Given: $\text{KB} = \{P, (P \rightarrow (Q \land R))\}$.
Show with resolution that $\text{KB} \models (R \lor S)$.

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Step 1: Reduce logical consequence to unsatisfiability.

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Step 1: Reduce logical consequence to unsatisfiability.

$\mathrm{KB} \models (R \vee S)$ iff $\mathrm{KB} \cup \{\neg(R \vee S)\}$ is unsatisfiable.

Thus, consider
$\mathrm{KB}' = \mathrm{KB} \cup \{\neg(R \vee S)\} = \{P, (P \rightarrow (Q \wedge R)), \neg(R \vee S)\}$.

...

Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\text{KB}' = \{P, (P \rightarrow (Q \wedge R)), \neg(R \vee S)\}$.

Step 2: Transform knowledge base into clause form (CNF).

Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\mathrm{KB}' = \{P, (P \rightarrow (Q \land R)), \neg(R \lor S)\}$.

Step 2: Transform knowledge base into clause form (CNF).

- P
  $\rightsquigarrow$ Clauses: $\{P\}$
- $P \rightarrow (Q \land R)) \equiv (\neg P \lor (Q \land R)) \equiv ((\neg P \lor Q) \land (\neg P \lor R))$
  $\rightsquigarrow$ Clauses: $\{\neg P, Q\}, \{\neg P, R\}$
- $\neg(R \lor S) \equiv (\neg R \land \neg S)$
  $\rightsquigarrow$ Clauses: $\{\neg R\}, \{\neg S\}$

Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\mathrm{KB}' = \{P, (P \rightarrow (Q \wedge R)), \neg(R \vee S)\}$.

Step 2: Transform knowledge base into clause form (CNF).

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$\Delta = \{\{P\}, \{\neg P, Q\}, \{\neg P, R\}, \{\neg R\}, \{\neg S\}\}$

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Proof by Resolution: Example (continued)

Proof by Resolution for Testing a Logical Consequence: Example

$\Delta = \{\{P\}, \{\neg P, Q\}, \{\neg P, R\}, \{\neg R\}, \{\neg S\}\}$

Step 3: Derive empty clause $\square$ with resolution.

- $C_1 = \{P\}$ (from Δ)
- $C_2 = \{\neg P, Q\}$ (from Δ)
- $C_3 = \{\neg P, R\}$ (from Δ)
- $C_4 = \{\neg R\}$ (from Δ)
- $C_5 = \{Q\}$ (from C_1 und C_2)
- $C_6 = \{\neg P\}$ (from C_3 und C_4)
- $C_7 = \square$ (from C_1 und C_6)

Note: There are shorter proofs. (For example?)

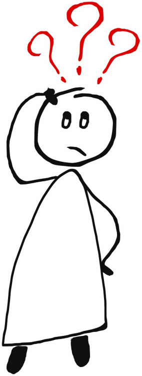
Another Example

Another Example for Resolution

Show with resolution, that KB $\models$ DrinkBeer, where

$$\begin{aligned}\text{KB} = \{&(\neg\text{DrinkBeer} \rightarrow \text{EatFish}),\\ &((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg\text{EatIceCream}),\\ &((\text{EatIceCream} \vee \neg\text{DrinkBeer}) \rightarrow \neg\text{EatFish})\}.\end{aligned}$$

Questions



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Summary

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- A logical consequence KB $\models \varphi$ allows to conclude that KB implies φ based on the semantics.
- A correct calculus supports such conclusions on the basis of purely syntactical derivations KB $\vdash \varphi$.
- Complete calculi often not necessary: For many questions refutation-completeness is sufficient.
- The resolution calculus is correct and refutation-complete.

Further Topics

There are many aspects of propositional logic that we do not cover in this course.

- resolution strategies to make resolution as efficient as possible in practice,
- other proof systems, as for example tableaux proofs,
- algorithms for model construction, such as the Davis-Putnam-Logemann-Loveland (DPLL) algorithm.

  $\rightarrow$ Foundations of AI course