Theory of Computer Science

B2. Propositional Logic II

Gabriele Röger

University of Basel

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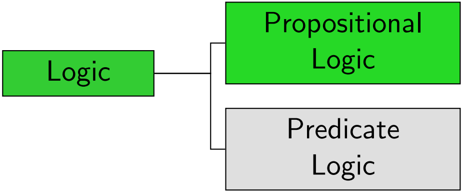
Logic: Overview



The Story So Far

- propositional logic based on atomic propositions
- syntax: which formulas are well-formed?
- semantics: when is a formula true?
- interpretations: important basis of semantics
- satisfiability and validity: important properties of formulas
- truth tables: systematically consider all interpretations
- equivalences: describe when formulas are semantically indistinguishable

Simplified Notation

Parentheses

Associativity:

$$((\varphi \land \psi) \land \chi) \equiv (\varphi \land (\psi \land \chi))$$
$$((\varphi \lor \psi) \lor \chi) \equiv (\varphi \lor (\psi \lor \chi))$$

- Placement of parentheses for a conjunction of conjunctions does not influence whether an interpretation is a model.
- ditto for disjunctions of disjunctions

$\rightarrow$ can omit parentheses and treat this as if parentheses placed arbitrarily

- Example: $(A_1 \land A_2 \land A_3 \land A_4)$ instead of $((A_1 \land (A_2 \land A_3)) \land A_4)$
- Example: $(\neg A \lor (B \land C) \lor D)$ instead of $((\neg A \lor (B \land C)) \lor D)$

Parentheses

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$$((\varphi \land \psi) \lor \chi) \not\equiv (\varphi \land (\psi \lor \chi))$$

What should $\varphi \land \psi \lor \chi$ mean?

Placement of Parentheses by Convention

Often parentheses can be dropped in specific cases
and an implicit placement is assumed:

- $\neg$ binds more strongly than $\wedge$
- $\wedge$ binds more strongly than $\vee$
- $\vee$ binds more strongly than $\rightarrow$ or $\leftrightarrow$

$\rightarrow$ cf. PEMDAS/“Punkt vor Strich”

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- often harder to read
- error-prone

$\rightarrow$ not used in this course

Short Notations for Conjunctions and Disjunctions

Short notation for addition:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

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Analogously (possible because of commutativity of $\wedge$ and $\vee$):

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$$\Big(\bigvee_{i=1}^{n} \varphi_i\Big) = (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$$

$$\Big(\bigwedge_{\varphi \in X} \varphi\Big) = (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$$

$$\Big(\bigvee_{\varphi \in X} \varphi\Big) = (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$$

$$\text{for } X = \{\varphi_1, \ldots, \varphi_n\}$$

Short Notation: Corner Cases

Is $\mathcal{I} \models \psi$ true for

$$\psi = (\bigwedge_{\varphi \in X} \varphi) \text{ and } \psi = (\bigvee_{\varphi \in X} \varphi)$$

if $X = \emptyset$ or $X = \{\chi\}$?

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convention:

- $\left(\bigwedge_{\varphi \in \emptyset} \varphi\right)$ is tautology.
- $\left(\bigvee_{\varphi \in \emptyset} \varphi\right)$ is unsatisfiable.
- $\left(\bigwedge_{\varphi \in \{\chi\}} \varphi\right) = \left(\bigvee_{\varphi \in \{\chi\}} \varphi\right) = \chi$

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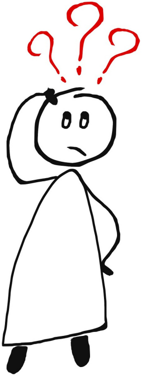
$\rightsquigarrow$ Why?

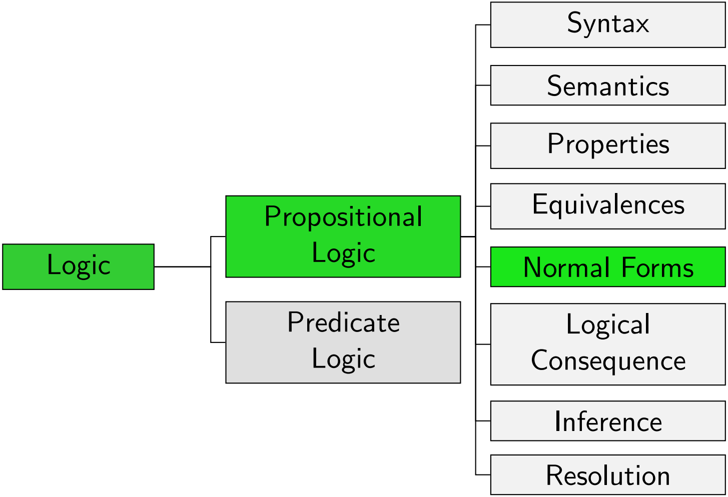
Questions



Questions?

Normal Forms

Logic: Overview



Why Normal Forms?

- A normal form is a representation with certain syntactic restrictions.
- condition for reasonable normal form: every formula must have a logically equivalent formula in normal form
- advantages:
  - can restrict proofs to formulas in normal form
  - can define algorithms only for formulas in normal form

German: Normalform

Literals, Clauses and Monomials

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The terms clause and monomial are also used for the corner case with only one literal.

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German: Literal, Klausel, Monom

Terminology: Examples

Examples

- $(\neg Q \wedge R)$
- $(P \vee \neg Q)$
- $((P \vee \neg Q) \wedge P)$
- $\neg P$
- $(P \rightarrow Q)$

- $(P \vee P)$
- $\neg\neg P$

Terminology: Examples

Examples

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Terminology: Examples

Examples

- $(\neg Q \wedge R)$ is a monomial
- $(P \vee \neg Q)$ is a clause
- $((P \vee \neg Q) \wedge P)$ is neither literal nor clause nor monomial
- $\neg P$
- $(P \rightarrow Q)$
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  (but $(\neg P \vee Q)$ is a clause!)
- $(P \vee P)$
- $\neg\neg P$

Terminology: Examples

Examples

- $(\neg Q \land R)$ is a monomial
- $(P \lor \neg Q)$ is a clause
- $((P \lor \neg Q) \land P)$ is neither literal nor clause nor monomial
- $\neg P$ is a literal, a clause and a monomial
- $(P \rightarrow Q)$ is neither literal nor clause nor monomial
  (but $(\neg P \lor Q)$ is a clause!)
- $(P \lor P)$ is a clause, but not a literal or monomial
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- $\neg\neg P$ is neither literal nor clause nor monomial

Conjunctive Normal Form

Definition (Conjunctive Normal Form)

A formula is in conjunctive normal form (CNF)
if it is a conjunction of clauses, i. e., if it has the form

$$\left(\bigwedge_{i=1}^{n}\left(\bigvee_{j=1}^{m_i} L_{ij}\right)\right)$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the L_{ij} are literals.

German: konjunktive Normalform (KNF)

Example

$((\neg P \lor Q) \land R \land (P \lor \neg S))$ is in CNF.

Disjunctive Normal Form

Definition (Disjunctive Normal Form)

A formula is in disjunctive normal form (DNF)
if it is a disjunction of monomials, i. e., if it has the form

$$\left(\bigvee_{i=1}^{n}\left(\bigwedge_{j=1}^{m_i} L_{ij}\right)\right)$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the L_{ij} are literals.

German: disjunktive Normalform (DNF)

Example

$((\neg P \land Q) \lor R \lor (P \land \neg S))$ is in DNF.

CNF and DNF: Examples

Examples

- $((P \vee \neg Q) \wedge P)$
- $((R \vee Q) \wedge P \wedge (R \vee S))$
- $(P \vee (\neg Q \wedge R))$
- $((P \vee \neg Q) \rightarrow P)$
- P

CNF and DNF: Examples

Examples

- $((P \vee \neg Q) \wedge P)$ is in CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$
- $(P \vee (\neg Q \wedge R))$
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CNF and DNF: Examples

Examples

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CNF and DNF: Examples

Examples

- $((P \vee \neg Q) \wedge P)$ is in CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$ is in CNF
- $(P \vee (\neg Q \wedge R))$ is in DNF
- $((P \vee \neg Q) \rightarrow P)$
- P

CNF and DNF: Examples

Examples

- $((P \lor \neg Q) \land P)$ is in CNF
- $((R \lor Q) \land P \land (R \lor S))$ is in CNF
- $(P \lor (\neg Q \land R))$ is in DNF
- $((P \lor \neg Q) \rightarrow P)$ is neither in CNF nor in DNF
- P

CNF and DNF: Examples

Examples

- $((P \vee \neg Q) \wedge P)$ is in CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$ is in CNF
- $(P \vee (\neg Q \wedge R))$ is in DNF
- $((P \vee \neg Q) \rightarrow P)$ is neither in CNF nor in DNF
- P is in CNF and in DNF

Construction of CNF (and DNF)

Algorithm to Construct CNF

1. Replace abbreviations $\rightarrow$ and $\leftrightarrow$ by their definitions (($\rightarrow$)-elimination and ($\leftrightarrow$)-elimination).
   $\rightsquigarrow$ formula structure: only $\vee$, $\wedge$, $\neg$
2. Move negations inside using De Morgan and double negation.
   $\rightsquigarrow$ formula structure: only $\vee$, $\wedge$, literals
3. Distribute $\vee$ over $\wedge$ with distributivity (strictly speaking also with commutativity).
   $\rightsquigarrow$ formula structure: CNF
4. optionally: Simplify the formula at the end or at intermediate steps (e. g., with idempotence).

Note: For DNF, distribute $\wedge$ over $\vee$ instead.

Constructing CNF: Example

Construction of Conjunctive Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

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Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$\varphi \equiv (\neg((P \wedge \neg Q) \vee R) \vee P \vee \neg(S \vee T)) \qquad \text{[Step 1]}$$

Constructing CNF: Example

Construction of Conjunctive Normal Form

Given: $\varphi = (((P \land \neg Q) \lor R) \rightarrow (P \lor \neg(S \lor T)))$

$$\begin{aligned} \varphi &\equiv (\neg((P \land \neg Q) \lor R) \lor P \lor \neg(S \lor T)) && \text{[Step 1]} \\ &\equiv ((\neg(P \land \neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]} \end{aligned}$$

Constructing CNF: Example

Construction of Conjunctive Normal Form

Given: $\varphi = (((P \land \neg Q) \lor R) \to (P \lor \neg(S \lor T)))$

$$\begin{aligned}
\varphi &\equiv (\neg((P \land \neg Q) \lor R) \lor P \lor \neg(S \lor T)) && \text{[Step 1]} \\
&\equiv ((\neg(P \land \neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]} \\
&\equiv (((\neg P \lor \neg\neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]}
\end{aligned}$$

Constructing CNF: Example

Construction of Conjunctive Normal Form

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&\equiv (((\neg P \vee \neg\neg Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) && \text{[Step 2]}\\
&\equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) && \text{[Step 2]}
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Constructing CNF: Example

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&\equiv ((\neg P \lor Q \lor P \lor (\neg S \land \neg T)) \land \\
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\end{aligned}
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&\qquad (\neg R \lor P \lor (\neg S \land \neg T))) && \text{[Step 3]}\\
&\equiv (\neg R \lor P \lor (\neg S \land \neg T)) && \text{[Step 4]}
\end{aligned}
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&\equiv ((\neg R \lor P \lor \neg S) \land (\neg R \lor P \lor \neg T)) && \text{[Step 3]}
\end{aligned}
$$

Construct DNF: Example

Construction of Disjunctive Normal Form

Given: $\varphi = (((P \land \neg Q) \lor R) \to (P \lor \neg(S \lor T)))$

$$
\begin{aligned}
\varphi &\equiv (\neg((P \land \neg Q) \lor R) \lor P \lor \neg(S \lor T)) && \text{[Step 1]} \\
&\equiv ((\neg(P \land \neg Q) \land \neg R) \lor P \lor \neg(S \lor T)) && \text{[Step 2]} \\
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Existence of an Equivalent Formula in Normal Form

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Theorem

For every formula φ there is a logically equivalent formula in CNF and a logically equivalent formula in DNF.

- "There is a" always means "there is at least one".
  Otherwise we would write "there is exactly one".
- Intuition: algorithm to construct normal form works with any given formula and only uses equivalence rewriting.
- actual proof would use induction over structure of formula

Size of Normal Forms

- In the worst case, a logically equivalent formula in CNF or DNF can be exponentially larger than the original formula.
- Example: for $(x_1 \vee y_1) \wedge \cdots \wedge (x_n \vee y_n)$ there is no smaller logically equivalent formula in DNF than:

$$\bigvee_{S \in \mathcal{P}(\{1,\ldots,n\})} \left(\bigwedge_{i \in S} x_i \wedge \bigwedge_{i \in \{1,\ldots,n\} \setminus S} y_i \right)$$

- As a consequence, the construction of the CNF/DNF formula can take exponential time.

More Theorems

Theorem

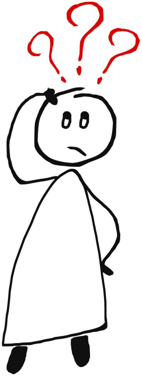
A formula in CNF is a tautology iff every clause is a tautology.

Theorem

A formula in DNF is satisfiable iff at least one of its monomials is satisfiable.

$\rightsquigarrow$ both proved easily with semantics of propositional logic

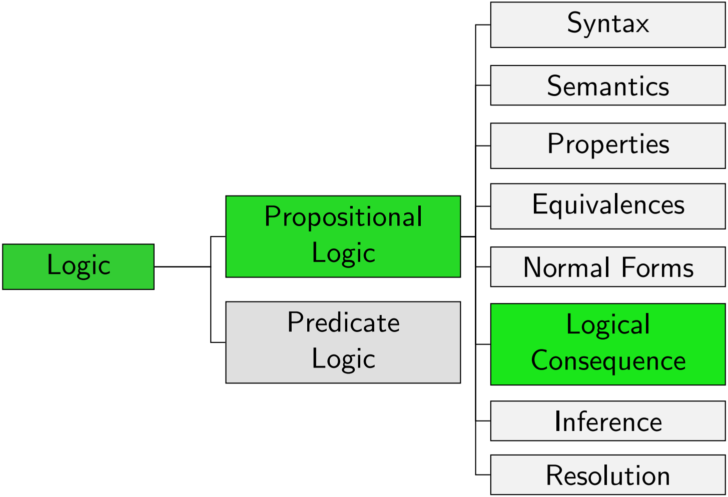
Questions



Questions?

Logical Consequences

Logic: Overview



Knowledge Bases: Example

If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

$$\begin{aligned} \mathrm{KB} = \{&(\neg \mathrm{DrinkBeer} \rightarrow \mathrm{EatFish}),\\ &((\mathrm{EatFish} \land \mathrm{DrinkBeer}) \rightarrow \neg \mathrm{EatIceCream}),\\ &((\mathrm{EatIceCream} \lor \neg \mathrm{DrinkBeer}) \rightarrow \neg \mathrm{EatFish})\} \end{aligned}$$

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Models for Sets of Formulas

Definition (Model for Knowledge Base)

Let KB be a knowledge base over A,
i. e., a set of propositional formulas over A.

A truth assignment $\mathcal{I}$ for A is a model for KB (written: $\mathcal{I} \models \text{KB}$) if $\mathcal{I}$ is a model for every formula $\varphi \in \text{KB}$.

German: Wissensbasis, Modell

Properties of Sets of Formulas

A knowledge base KB is

- satisfiable if KB has at least one model
- unsatisfiable if KB is not satisfiable
- valid (or a tautology) if every interpretation is a model for KB
- falsifiable if KB is no tautology

German: erfüllbar, unerfüllbar, gültig, gültig/eine Tautologie, falsifizierbar

Example I

Which of the properties does $KB = \{(A \land \neg B), \neg(B \lor A)\}$ have?

Example I

Which of the properties does $\mathrm{KB} = \{(\mathrm{A} \land \lnot \mathrm{B}), \lnot(\mathrm{B} \lor \mathrm{A})\}$ have?

KB is unsatisfiable:
For every model $\mathcal{I}$ with $\mathcal{I} \models (\mathrm{A} \land \lnot \mathrm{B})$ we have $\mathcal{I}(\mathrm{A}) = 1$.
This means $\mathcal{I} \models (\mathrm{B} \lor \mathrm{A})$ and thus $\mathcal{I} \not\models \lnot(\mathrm{B} \lor \mathrm{A})$.

Example I

Which of the properties does $\mathsf{KB} = \{(\mathsf{A} \land \neg\mathsf{B}), \neg(\mathsf{B} \lor \mathsf{A})\}$ have?

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This means $\mathcal{I} \models (\mathsf{B} \lor \mathsf{A})$ and thus $\mathcal{I} \not\models \neg(\mathsf{B} \lor \mathsf{A})$.

This directly implies that KB is falsifiable, not satisfiable and no tautology.

Example II

Which of the properties does

$$\begin{aligned} \mathsf{KB} = \{&(\neg \mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}),\\ &((\mathsf{EatFish} \wedge \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatIceCream}),\\ &((\mathsf{EatIceCream} \vee \neg \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatFish})\} \text{ have?} \end{aligned}$$

Example II

Which of the properties does

$$\begin{aligned}\mathrm{KB} = \{&(\neg\mathrm{DrinkBeer} \rightarrow \mathrm{EatFish}),\\ &((\mathrm{EatFish} \land \mathrm{DrinkBeer}) \rightarrow \neg\mathrm{EatIceCream}),\\ &((\mathrm{EatIceCream} \lor \neg\mathrm{DrinkBeer}) \rightarrow \neg\mathrm{EatFish})\}\text{ have?}\end{aligned}$$

- satisfiable, e. g. with $\mathcal{I} = \{\mathrm{EatFish} \mapsto 1, \mathrm{DrinkBeer} \mapsto 1, \mathrm{EatIceCream} \mapsto 0\}$
- thus not unsatisfiable
- falsifiable, e. g. with $\mathcal{I} = \{\mathrm{EatFish} \mapsto 0, \mathrm{DrinkBeer} \mapsto 0, \mathrm{EatIceCream} \mapsto 1\}$
- thus not valid

Logical Consequences: Motivation

What's the secret of your long life?

I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Claim: the woman drinks beer to every meal.

How can we prove this?

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut/FreeDigitalPhotos.net

Logical Consequences

Definition (Logical Consequence)

Let KB be a set of formulas and φ a formula.

We say that KB **logically implies** φ (written as $KB \models \varphi$) if **all models** of KB are also models of φ.

also: KB **logically entails** φ, φ **logically follows** from KB, φ is a **logical consequence** of KB

German: KB impliziert φ logisch, φ folgt logisch aus KB, φ ist logische Konsequenz von KB

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Attention: the symbol $\models$ is "overloaded": $\text{KB} \models \varphi$ vs. $\mathcal{I} \models \varphi$.

What if KB is unsatisfiable or the empty set?

Logical Consequences: Example

Let $\varphi = \text{DrinkBeer}$ and

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Show: $\text{KB} \models \varphi$

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Show: $\text{KB} \models \varphi$

Proof sketch.

Proof by contradiction: assume $\mathcal{I} \models \text{KB}$, but $\mathcal{I} \not\models \text{DrinkBeer}$.
Then it follows that $\mathcal{I} \models \neg\text{DrinkBeer}$.
Because $\mathcal{I}$ is a model of KB, we also have
$\mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$ and thus $\mathcal{I} \models \text{EatFish}$. (Why?)
With an analogous argumentation starting from
$\mathcal{I} \models ((\text{EatIceCream} \lor \neg\text{DrinkBeer}) \rightarrow \neg\text{EatFish})$
we get $\mathcal{I} \models \neg\text{EatFish}$ and thus $\mathcal{I} \not\models \text{EatFish}$. $\rightsquigarrow$ Contradiction!

Important Theorems about Logical Consequences

Theorem (Deduction Theorem)

$\mathrm{KB} \cup \{\varphi\} \models \psi$ *iff* $\mathrm{KB} \models (\varphi \rightarrow \psi)$

German: Deduktionssatz

Theorem (Contraposition Theorem)

$\mathrm{KB} \cup \{\varphi\} \models \neg\psi$ *iff* $\mathrm{KB} \cup \{\psi\} \models \neg\varphi$

German: Kontrapositionssatz

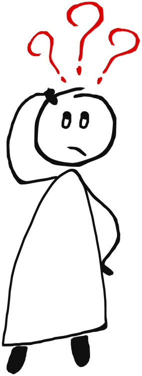
Theorem (Contradiction Theorem)

$\mathrm{KB} \cup \{\varphi\}$ *is unsatisfiable iff* $\mathrm{KB} \models \neg\varphi$

German: Widerlegungssatz

(without proof)

Questions



Questions?

Summary

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- CNF: formula is a conjunction of clauses
- DNF: formula is a disjunction of monomials
- every formula has equivalent formulas in DNF and in CNF
- knowledge base: set of formulas describing given information; satisfiable, valid etc. used like for individual formulas
- logical consequence KB $\models \varphi$ means that φ is true whenever (= in all models where) KB is true