# Theory of Computer Science

## A2. Mathematical Foundations

Gabriele Röger

University of Basel

February 18, 2019



# Theory of Computer Science

February 18, 2019 — A2. Mathematical Foundations







# A2.1 Sets, Tuples, Relations





## Sets

- set: unordered collection of distinguishable objects; each object contained at most once
- notations:
  - explicit, listing all elements, e. g. $A = \{1, 2, 3\}$
  - implicit, specifying a property characterizing all elements, e. g. $A = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 3\}$
  - implicit, as a sequence with dots, e. g. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- $e \in M$: $e$ is in set $M$ (an element of the set)
- $e \notin M$: $e$ is not in set $M$
- empty set $\emptyset = \{\}$
- cardinality $|M|$ of a finite set $M$: number of elements in $M$

German: Menge, Element, leere Menge, Mächtigkeit/Kardinalität

## Sets

- $A \subseteq B$: $A$ is a subset of $B$,
  i. e., every element of $A$ is an element of $B$
- $A \subset B$: $A$ is a strict subset of $B$,
  i. e., $A \subseteq B$ and $A \neq B$.
- power set $\mathcal{P}(M)$: set of all subsets of $M$
  e. g., $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

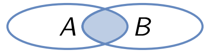
German: Teilmenge, echte Teilmenge, Potenzmenge



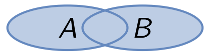


## Set Operations

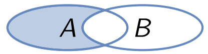
- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



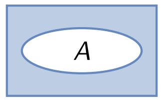

- union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$




- difference $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$




- complement $\overline{A} = B \setminus A$, where $A \subseteq B$ and
  $B$ is the set of all considered objects (in a given context)




German: Schnitt, Vereinigung, Differenz, Komplement





## Tuples

- $k$-tuple: ordered sequence of $k$ objects
- written $(o_1, \dots, o_k)$ or $\langle o_1, \dots, o_k \rangle$
- unlike sets, order matters ($\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$)
- objects may occur multiple times in a tuple
- objects contained in tuples are called components
- terminology:
  - $k = 2$: (ordered) pair
  - $k = 3$: triple
  - more rarely: quadruple, quintuple, sextuple, septuple, ...
- if $k$ is clear from context (or does not matter),
  often just called tuple

German: $k$-Tupel, Komponente, Paar, Tripel





## Cartesian Product

- for sets $M_1, M_2, \dots, M_n$, the Cartesian product
  $M_1 \times \dots \times M_n$ is the set
  $M_1 \times \dots \times M_n = \{\langle o_1, \dots, o_n \rangle \mid o_1 \in M_1, \dots, o_n \in M_n\}$.
- Example: $M_1 = \{a, b, c\}, M_2 = \{1, 2\}$,
  $M_1 \times M_2 = \{\langle a, 1 \rangle, \langle a, 2 \rangle, \langle b, 1 \rangle, \langle b, 2 \rangle, \langle c, 1 \rangle, \langle c, 2 \rangle\}$
- special case: $M^k = M \times \dots \times M$ ($k$ times)
- example with $M = \{1, 2\}$:
  $M^2 = \{\langle 1, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 2, 2 \rangle\}$

German: kartesisches Produkt

## Relations

- an $n$-ary relation $R$ over the sets $M_1, \dots, M_n$ is a subset of their Cartesian product: $R \subseteq M_1 \times \dots \times M_n$.
- example with $M = \{1, 2\}$: $R_{\leq} \subseteq M^2$ as $R_{\leq} = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$

German: ($n$-stellige) Relation





## Quiz

Consider $S = \mathcal{P}(\{1, 2\}) \times \{a, b\}$.

1. Write down three different elements of $S$.
2. What is $|S|$?





# A2.2 Functions





## Functions

**Definition (Total Function)**

A (total) function $f : D \to C$ (with sets $D, C$) maps every value of its domain $D$ to exactly one value of its codomain $C$.

German: (totale) Funktion, Definitionsbereich, Wertebereich

**Example**

- $square : \mathbb{Z} \to \mathbb{Z}$ with $square(x) = x^2$
- $add : \mathbb{N}_0^2 \to \mathbb{N}_0$ with $add(x, y) = x + y$
- $add_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}$ with $add_{\mathbb{R}}(x, y) = x + y$

## Functions: Example

**Example**

Let $Q = \{q_0, q_1, q_2, q_e\}$ and $\Gamma = \{0, 1, \Box\}$.

Define $\delta : (Q \setminus \{q_e\}) \times \Gamma \to Q \times \Gamma \times \{L, R, N\}$ by

| $\delta$ | 0 | 1 | $\Box$ |
|---|---|---|---|
| $q_0$ | $\langle q_0, 0, R\rangle$ | $\langle q_0, 1, R\rangle$ | $\langle q_1, \Box, L\rangle$ |
| $q_1$ | $\langle q_2, 1, L\rangle$ | $\langle q_1, 0, L\rangle$ | $\langle q_e, 1, N\rangle$ |
| $q_2$ | $\langle q_2, 0, L\rangle$ | $\langle q_2, 1, L\rangle$ | $\langle q_e, \Box, R\rangle$ |

Then, e. g., $\delta(q_0, 1) = \langle q_0, 1, R\rangle$





## Partial Functions

**Definition (Partial Function)**

A partial function $f : X \to_p Y$ maps every value in $X$ to at most one value in $Y$.

If $f$ does not map $x \in X$ to any value in $Y$, then $f$ is undefined for $x$.

German: partielle Funktion

**Example**

$f : \mathbb{N}_0 \times \mathbb{N}_0 \to_p \mathbb{N}_0$ with

$$f(x, y) = \begin{cases} x - y & \text{if } y \leq x \\ \text{undefined} & \text{otherwise} \end{cases}$$





# A2.3 Summary





## Summary

- sets: unordered, contain every element at most once
- tuples: ordered, can contain the same object multiple times
- Cartesian product: $M_1 \times \cdots \times M_n$ set of all $n$-tuples where the $i$-th component is in $M_i$
- function $f : X \to Y$ maps every value in $X$ to exactly one value in $Y$
- partial function $g : X \to_p Y$ may be undefined for some values in $X$