Theory of Computer Science

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Exercise Sheet 3 — Solutions

Exercise 3.1 (Refutation Theorem; 1.5 Points)

Prove the refutation theorem, that is, show for any set of formulas KB and any formula φ that

$$\mathrm{KB} \cup \{\varphi\} \text{ is unsatisfiable if and only if } \mathrm{KB} \models \neg\varphi.$$

Solution:

"$\Rightarrow$": If $\mathrm{KB} \cup \{\varphi\}$ is unsatisfiable then there is no interpretation $\mathcal{I}$ with $\mathcal{I} \models \mathrm{KB}$ and $\mathcal{I} \models \varphi$. Hence for every $\mathcal{I}$ with $\mathcal{I} \models \mathrm{KB}$ it holds that $\mathcal{I} \not\models \varphi$ and we conclude that $\mathrm{KB} \models \neg\varphi$.

"$\Leftarrow$": If $\mathrm{KB} \models \neg\varphi$ then it holds for all $\mathcal{I}$ with $\mathcal{I} \models \mathrm{KB}$ that $\mathcal{I} \models \neg\varphi$ and hence $\mathcal{I} \not\models \varphi$. Therefore there is no interpretation with $\mathcal{I} \models \mathrm{KB} \cup \{\varphi\}$, so $\mathrm{KB} \cup \{\varphi\}$ is unsatisfiable.

Exercise 3.2 (Correctness of the Resolution Calculus; 1.5 Points)

Prove the correctness of the resolution rule

$$\frac{C_1 \cup \{L\},\ C_2 \cup \{\neg L\}}{C_1 \cup C_2},$$

by showing that for all interpretations $\mathcal{I}$ with $\mathcal{I} \models \bigvee_{\ell \in C_1 \cup \{L\}} \ell$ and $\mathcal{I} \models \bigvee_{\ell \in C_2 \cup \{\neg L\}} \ell$ it holds that $I \models \bigvee_{\ell \in C_1 \cup C_2} \ell$.

Solution:

Let $\mathcal{I}$ be a model of $\bigvee_{\ell \in C_1 \cup \{L\}} \ell$ and of $\bigvee_{\ell \in C_2 \cup \{\neg L\}} \ell$. We make a case distinction:

Case 1 ($\mathcal{I} \models L$)*:* In this case we can conclude from $\mathcal{I} \models \bigvee_{\ell \in C_2 \cup \{\neg L\}} \ell$ that $\mathcal{I} \models \bigvee_{\ell \in C_2} \ell$. This implies that $I \models \bigvee_{\ell \in C_1 \cup C_2} \ell$.

Case 2 ($\mathcal{I} \not\models L$)*:* It follows from $\mathcal{I} \models \bigvee_{\ell \in C_1 \cup \{L\}} \ell$ that $\mathcal{I} \models \bigvee_{\ell \in C_1} \ell$ and hence $I \models \bigvee_{\ell \in C_1 \cup C_2} \ell$.

All steps were based on the semantics of the disjunction. Since the two cases cover all possible situations, we conclude that $I \models \bigvee_{\ell \in C_1 \cup C_2} \ell$.

Exercise 3.3 (Resolution Calculus; 3 Points)

Consider the following knowledge base

$$\begin{aligned}\mathrm{KB} = \{&(A \leftrightarrow \neg D), (\neg A \rightarrow (B \vee C)), ((A \rightarrow E) \wedge (B \vee C \vee F)), (E \rightarrow (F \rightarrow (B \vee C))),\\ &(C \rightarrow G), (G \rightarrow \neg C)\}.\end{aligned}$$

Use the resolution calculus to show that $\mathrm{KB} \models (B \wedge \neg C)$.

Solution:

In order to show that $KB \models (B \wedge \neg C)$, we prove that $KB' = KB \cup \{\neg(B \wedge \neg C)\}$ is unsatisfiable. Since we want to apply resolution we first need to transform KB' into conjunctive normal form:

Formulas (and Equivalences)	Clauses
$(A \leftrightarrow \neg D)$ $\equiv ((A \rightarrow \neg D) \wedge (\neg D \rightarrow A))$ $\equiv ((\neg A \vee \neg D) \wedge (\neg\neg D \vee A))$ $\equiv ((\neg A \vee \neg D) \wedge (D \vee A))$	$\{\neg A, \neg D\}$ $\{A, D\}$
$(\neg A \rightarrow (B \vee C))$ $\equiv (\neg\neg A \vee (B \vee C))$ $\equiv (A \vee B \vee C)$	$\{A, B, C\}$
$((A \rightarrow E) \wedge (B \vee C \vee F))$ $\equiv ((\neg A \vee E) \wedge (B \vee C \vee F))$	$\{\neg A, E\}$ $\{B, C, F\}$
$(E \rightarrow (F \rightarrow (B \vee C)))$ $\equiv (\neg E \vee (\neg F \vee (B \vee C)))$ $\equiv (\neg E \vee \neg F \vee B \vee C)$	$\{B, C, \neg E, \neg F\}$
$(C \rightarrow G) \equiv (\neg C \vee G)$	$\{\neg C, G\}$
$(G \rightarrow \neg C) \equiv (\neg G \vee \neg C)$	$\{\neg C, \neg G\}$
$\neg(B \wedge \neg C) \equiv (\neg B \vee C)$	$\{\neg B, C\}$

We need to derive the emtpy clause $\square$ from the following set of clauses Δ:

$$\Delta = \{\{\neg A, \neg D\}, \{A, D\}, \{A, B, C\}, \{\neg A, E\}, \{B, C, F\}, \\ \{B, C, \neg E, \neg F\}, \{\neg C, G\}, \{\neg C, \neg G\}, \{\neg B, C\}\}.$$

One possible derivation:

$K_1 = \{\neg C, G\}$	from Δ
$K_2 = \{\neg C, \neg G\}$	from Δ
$K_3 = \{\neg C\}$	from K_1 and K_2
$K_4 = \{A, B, C\}$	from Δ
$K_5 = \{A, B\}$	from K_3 and K_4
$K_6 = \{C, \neg B\}$	from Δ
$K_7 = \{\neg B\}$	from K_3 and K_6
$K_8 = \{A\}$	from K_5 and K_7
$K_9 = \{\neg A, E\}$	from Δ
$K_{10} = \{E\}$	from K_8 and K_9
$K_{11} = \{B, C, \neg E, \neg F\}$	from Δ
$K_{12} = \{B, C, \neg F\}$	from K_{10} and K_{11}
$K_{13} = \{C, \neg F\}$	from K_7 and K_{12}
$K_{14} = \{\neg F\}$	from K_3 and K_{13}
$K_{15} = \{B, C, F\}$	from Δ
$K_{16} = \{C, F\}$	from K_7 and K_{15}
$K_{17} = \{C\}$	from K_{14} and K_{16}
$K_{18} = \square$	fromK_3 and K_{17}

With the contradiction theorem we can conclude that $KB \models (B \wedge \neg C)$.

Exercise 3.4 (Predicate Logic; 3 Points)

Consider the following predicate logic formula φ with the signature $\langle \{x, y\}, \{\mathrm{c}\}, \{\mathrm{f}, \mathrm{g}\}, \{\mathrm{P}\} \rangle$.

$$\varphi = (\neg \mathrm{P}(\mathrm{c}) \wedge \forall x \exists y\, ((\mathrm{f}(y) = \mathrm{g}(x)) \wedge \mathrm{P}(y)))$$

Specify a model $\mathcal{I}$ of φ with $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ and $\mathcal{U} = \{u_1, u_2, u_3\}$. Prove that $\mathcal{I} \models \varphi$. Why is no variable assignment α required to specify a model of φ?

Solution:

The following interpretation $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ is a model of φ.

$$\begin{aligned}
&U = \{u_1, u_2, u_3\} \\
&\mathrm{c}^{\mathcal{I}} = u_1 \\
&\mathrm{f}^{\mathcal{I}} = \mathrm{g}^{\mathcal{I}} = \{u_1 \mapsto u_3, u_2 \mapsto u_3, u_3 \mapsto u_3\} \\
&\mathrm{P}^{\mathcal{I}} = \{u_3\}
\end{aligned}$$

For every $u \in U$ we define $\alpha_u = x \mapsto u, y \mapsto u_3$.

$$\begin{aligned}
\text{By definition we have } \mathrm{f}(y)^{\mathcal{I},\alpha_u} &= \mathrm{f}^{\mathcal{I}}(y^{\mathcal{I},\alpha_u}) = \mathrm{f}^{\mathcal{I}}(\alpha_u(y)) = \mathrm{f}^{\mathcal{I}}(u_3) = u_3 \\
\text{and } \mathrm{g}(x)^{\mathcal{I},\alpha_u} &= \mathrm{g}^{\mathcal{I}}(x^{\mathcal{I},\alpha_u}) = \mathrm{g}^{\mathcal{I}}(\alpha_u(x)) = \mathrm{g}^{\mathcal{I}}(u) = u_3 \\
\text{and thus } \mathrm{f}(y)^{\mathcal{I},\alpha_u} &= \mathrm{g}(x)^{\mathcal{I},\alpha_u} \\
\text{It follows that } \mathcal{I}, \alpha_u &\models (\mathrm{f}(y) = \mathrm{g}(x)) \\
\text{By definition we also have } y^{\mathcal{I},\alpha_u} &= \alpha_u(y) = u_3 \in \mathrm{P}^{\mathcal{I}} \text{ so } \mathcal{I}, \alpha_u \models \mathrm{P}(y) \\
\text{Together with the line before that we get } \mathcal{I}, \alpha_u &\models ((\mathrm{f}(y) = \mathrm{g}(x)) \wedge \mathrm{P}(y))
\end{aligned}$$

For every variable assignment α we know that $\alpha[x := u][y := u_3] = \alpha_u$. Thus, for every α and every $u \in U$

$$\begin{aligned}
\mathcal{I}, \alpha[x := u][y := u_3] &\models ((\mathrm{f}(y) = \mathrm{g}(x)) \wedge \mathrm{P}(y)) \\
\mathcal{I}, \alpha[x := u] &\models \exists y\, ((\mathrm{f}(y) = \mathrm{g}(x)) \wedge \mathrm{P}(y)) \\
\mathcal{I}, \alpha &\models \forall x \exists y\, ((\mathrm{f}(y) = \mathrm{g}(x)) \wedge \mathrm{P}(y))
\end{aligned}$$

We also know $\mathrm{c}^{\mathcal{I},\alpha} = \mathrm{c}^{\mathcal{I}} = u_1 \notin \mathrm{P}^{\mathcal{I}}$ so $\mathcal{I}, \alpha \not\models \mathrm{P}(\mathrm{c})$ or $\mathcal{I}, \alpha \models \neg\mathrm{P}(\mathrm{c})$. Putting things together, we get

$$\mathcal{I}, \alpha \models (\neg\mathrm{P}(\mathrm{c}) \wedge \forall x \exists y\, ((\mathrm{f}(y) = \mathrm{g}(x)) \wedge \mathrm{P}(y))).$$

Since all variables are bound, the proof does not depend on the variable assignment α, so it is not required to specify a model.

Exercise 3.5 (Predicate logic; 1 Point)

Consider the formula φ over a signature with predicate symbols P (1-ary), Q (2-ary) and R (3-ary), the 1-ary function symbol f, the constant symbol c and the variable symbols x, y and z.

$$\varphi = (\forall x \exists y\, (\mathrm{P}(z) \rightarrow \mathrm{Q}(y, x)) \vee \neg\exists y\, \mathrm{R}(\mathrm{c}, x, \mathrm{f}(y)))$$

Mark all occurrences of free variables in φ. *Additionaly* specify the set of free variables of φ (without proof).

Solution:

$$\varphi = (\forall x \exists y\, (\mathrm{P}(\underline{z}) \rightarrow \mathrm{Q}(y, x)) \vee \neg\exists y\, \mathrm{R}(\mathrm{c}, \underline{x}, \mathrm{f}(y))).$$

$$\mathit{free}(\varphi) = \{x, z\}$$