Theory of Computer Science

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Exercise Sheet 2 — Solutions

Exercise 2.1 (Semantics; 0.5+0.5+1+1+1 Points)

Consider the propositional formula φ over $\{A, B, C, D, E, F\}$:

$$\varphi = ((F \lor ((\neg B \leftrightarrow ((C \land A) \rightarrow \neg B)) \lor (D \rightarrow E))) \rightarrow (A \rightarrow \neg F))$$

(a) How many lines would be needed for a truth table for φ?

Solution:

φ contains 6 atomic statements, thus a truth table for φ would need $2^6 = 64$ lines.

(b) Formula φ is an implication. Specify the truth table for the general implication formula $\varphi \rightarrow \psi$. Attention: You should **not** specify the truth table of φ.

Solution:

$\mathcal{I} \models \varphi$	$\mathcal{I} \models \psi$	$\mathcal{I} \models (\varphi \rightarrow \psi)$
No	No	Yes
No	Yes	Yes
Yes	No	No
Yes	Yes	Yes

(c) Specify a model $\mathcal{I}$ for φ and prove without truth table that $\mathcal{I} \models \varphi$.

Solution:

$$\mathcal{I} = \{A \mapsto 1, B \mapsto 1, C \mapsto 1, D \mapsto 1, E \mapsto 1, F \mapsto 0\}$$

To make the proof more concise, we define:

$$\psi = (F \lor ((\neg B \leftrightarrow ((C \land A) \rightarrow \neg B)) \lor (D \rightarrow E)))$$

From $\mathcal{I}(F) = 0$ follows $\mathcal{I} \not\models F$ and from this also $\mathcal{I} \models \neg F$. Thus $\mathcal{I} \models (A \rightarrow \neg F)$ holds (no matter whether $\mathcal{I} \models A$ holds or not). From this we conclude $\mathcal{I} \models (\psi \rightarrow (A \rightarrow \neg F))$ (no matter whether $\mathcal{I} \models \psi$ holds or not). The formula $(\psi \rightarrow (A \rightarrow \neg F))$ is φ, which means we have shown that $\mathcal{I} \models \varphi$.

(d) Specify an assignment $\mathcal{I}$ with $\mathcal{I} \not\models \varphi$ and prove that $\mathcal{I}$ has the desired property without a truth table.

Solution:

$$\mathcal{I} = \{A \mapsto 1, B \mapsto 1, C \mapsto 1, D \mapsto 1, E \mapsto 1, F \mapsto 1\}$$

To make the proof more concise, we define:

$$\psi = ((\neg B \leftrightarrow ((C \land A) \rightarrow \neg B)) \lor (D \rightarrow E))$$

From $\mathcal{I}(F) = 1$ follows $\mathcal{I} \models F$ and from this also $\mathcal{I} \models (F \lor \psi)$ (no matter whether $\mathcal{I} \models \psi$ holds or not).

From $\mathcal{I}(\mathrm{A}) = 1$ follows $\mathcal{I} \models \mathrm{A}$. From $\mathcal{I} \models \mathrm{F}$ follows $\mathcal{I} \not\models \neg\mathrm{F}$. From $\mathcal{I} \models \mathrm{A}$ together with $\mathcal{I} \not\models \neg\mathrm{F}$ we can conclude that $\mathcal{I} \not\models (\mathrm{A} \rightarrow \neg\mathrm{F})$.

From $\mathcal{I} \models (\mathrm{F} \vee \psi)$ and $\mathcal{I} \not\models (\mathrm{A} \rightarrow \neg\mathrm{F})$ follows $\mathcal{I} \not\models ((\mathrm{F} \vee \psi) \rightarrow (\mathrm{A} \rightarrow \neg\mathrm{F}))$. The formula $((\mathrm{F} \vee \psi) \rightarrow (\mathrm{A} \rightarrow \neg\mathrm{F}))$ is φ, which means we have shown that $\mathcal{I} \not\models \varphi$.

(e) Which of the properties *satisfiable*, *unsatisfiable*, *valid*, and *falsifiable* are true for φ? Justify your answer for each of the four properties.

Solution:

- Formula φ is satisfiable because the interpretation from part (c) is a model. Thus, it is not unsatisfiable.
- Formula φ is falsifiable because the interpretation from part (d) is no model. Thus, it is not valid.

Exercise 2.2 (Equivalences; 1.5+1.5 Points)

(a) Transform the following formula into CNF by applying the equivalence rules shown in the lecture. For each step, only apply one equivalence rule and also specify it.

$$\varphi = ((\mathrm{A} \rightarrow \mathrm{B}) \leftrightarrow \neg\mathrm{C})$$

Solution:

$$\begin{aligned}
\varphi &= ((\mathrm{A} \rightarrow \mathrm{B}) \leftrightarrow \neg\mathrm{C}) & \\
&\equiv ((\neg\mathrm{A} \vee \mathrm{B}) \leftrightarrow \neg\mathrm{C}) & (\rightarrow\text{-Elimination}) \\
&\equiv (((\neg\mathrm{A} \vee \mathrm{B}) \rightarrow \neg\mathrm{C}) \wedge (\neg\mathrm{C} \rightarrow (\neg\mathrm{A} \vee \mathrm{B}))) & (\leftrightarrow\text{-Elimination}) \\
&\equiv ((\neg(\neg\mathrm{A} \vee \mathrm{B}) \vee \neg\mathrm{C}) \wedge (\neg\mathrm{C} \rightarrow (\neg\mathrm{A} \vee \mathrm{B}))) & (\rightarrow\text{-Elimination}) \\
&\equiv ((\neg(\neg\mathrm{A} \vee \mathrm{B}) \vee \neg\mathrm{C}) \wedge (\neg\neg\mathrm{C} \vee (\neg\mathrm{A} \vee \mathrm{B}))) & (\rightarrow\text{-Elimination}) \\
&\equiv ((\neg(\neg\mathrm{A} \vee \mathrm{B}) \vee \neg\mathrm{C}) \wedge (\mathrm{C} \vee (\neg\mathrm{A} \vee \mathrm{B}))) & (\text{Double negation}) \\
&\equiv (((\neg\neg\mathrm{A} \wedge \neg\mathrm{B}) \vee \neg\mathrm{C}) \wedge (\mathrm{C} \vee (\neg\mathrm{A} \vee \mathrm{B}))) & (\text{De Morgan}) \\
&\equiv (((\mathrm{A} \wedge \neg\mathrm{B}) \vee \neg\mathrm{C}) \wedge (\mathrm{C} \vee (\neg\mathrm{A} \vee \mathrm{B}))) & (\text{Double negation}) \\
&\equiv ((\neg\mathrm{C} \vee (\mathrm{A} \wedge \neg\mathrm{B})) \wedge (\mathrm{C} \vee (\neg\mathrm{A} \vee \mathrm{B}))) & (\text{Commutativity}) \\
&\equiv (((\neg\mathrm{C} \vee \mathrm{A}) \wedge (\neg\mathrm{C} \vee \neg\mathrm{B})) \wedge (\mathrm{C} \vee (\neg\mathrm{A} \vee \mathrm{B}))) & (\text{Distributivity})
\end{aligned}$$

(b) Prove that the following formula is unsatisfiable by showing that $\varphi \equiv (\mathrm{A} \wedge \neg\mathrm{A})$ holds. Use the equivalence rules from the lecture, only apply one rule for each step and specify the applied rule.

$$\varphi = \neg((\mathrm{A} \wedge (\neg\mathrm{B} \rightarrow \mathrm{A})) \vee \neg\mathrm{A})$$

Solution:

$$\begin{aligned}
\varphi &= \neg((\mathrm{A} \wedge (\neg\mathrm{B} \rightarrow \mathrm{A})) \vee \neg\mathrm{A}) & \\
&\equiv \neg((\mathrm{A} \wedge (\neg\neg\mathrm{B} \vee \mathrm{A})) \vee \neg\mathrm{A}) & (\rightarrow\text{-Elimination}) \\
&\equiv \neg((\mathrm{A} \wedge (\mathrm{A} \vee \neg\neg\mathrm{B})) \vee \neg\mathrm{A}) & (\text{Commutativity}) \\
&\equiv \neg(\mathrm{A} \vee \neg\mathrm{A}) & (\text{Absorption}) \\
&\equiv (\neg\mathrm{A} \wedge \neg\neg\mathrm{A}) & (\text{De Morgan}) \\
&\equiv (\neg\mathrm{A} \wedge \mathrm{A}) & (\text{Double negation}) \\
&\equiv (\mathrm{A} \wedge \neg\mathrm{A}) & (\text{Commutativity})
\end{aligned}$$

Exercise 2.3 (Logical Consequence; 1.5+1.5 Points)

Consider the following formula set over $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$.

$$\mathrm{KB} = \{(\mathrm{A} \rightarrow \neg\mathrm{C}), (\mathrm{A} \vee \neg\mathrm{B}), (\neg\mathrm{A} \vee \mathrm{C})\}$$

(a) Does a model $\mathcal{I}$ of KB exist which is also a model for $\varphi = (\mathrm{A} \vee \mathrm{B})$? Prove your statement.

Solution:

No. We will prove this by contradiction: Assume there is a $\mathcal{I}$ with $\mathcal{I} \models \mathrm{KB}$ and $\mathcal{I} \models (\mathrm{A} \vee \mathrm{B})$. Then it holds that $\mathcal{I} \models \mathrm{A}$ or $\mathcal{I} \models \mathrm{B}$. We distinct those two cases:

Case 1 ($\mathcal{I} \models \mathrm{A}$): From $\mathcal{I} \models \mathrm{KB}$ follows that $\mathcal{I} \models (\mathrm{A} \rightarrow \neg\mathrm{C}) = (\neg\mathrm{A} \vee \neg\mathrm{C})$. From this we conclude $\mathcal{I} \models \neg\mathrm{A}$ or $\mathcal{I} \models \neg\mathrm{C}$. The first case cannot occur since from $\mathcal{I} \models \mathrm{A}$ follows $\mathcal{I} \not\models \neg\mathrm{A}$. Thus the second case ($\mathcal{I} \models \neg\mathrm{C}$) must hold.

But from $\mathcal{I} \models \mathrm{KB}$ also follows that $\mathcal{I} \models (\neg\mathrm{A} \vee \mathrm{C})$, which means $\mathcal{I} \models \neg\mathrm{A}$ or $\mathcal{I} \models \mathrm{C}$. But neither $\mathcal{I} \models \neg\mathrm{A}$ can hold (since $\mathcal{I} \models \mathrm{A}$ holds), nor $\mathcal{I} \models \mathrm{C}$ (since $\mathcal{I} \models \neg\mathrm{C}$ holds). Thus this case results in a contradiction.

Case 2 ($\mathcal{I} \not\models \mathrm{A}$): Since $\mathcal{I} \models (\mathrm{A} \vee \mathrm{B})$, it holds that $\mathcal{I} \models \mathrm{B}$ and therefore $\mathcal{I} \not\models \neg\mathrm{B}$. Since $\mathcal{I} \not\models \mathrm{A}$ and $I \not\models \neg\mathrm{B}$, we know that $\mathcal{I} \not\models (\mathrm{A} \vee \neg\mathrm{B})$. As $(\mathrm{A} \vee \neg\mathrm{B}) \in \mathrm{KB}$, this is a contradiction to $\mathcal{I} \models \mathrm{KB}$.

(b) Prove that all models $\mathcal{I}$ of KB are also models of $\varphi = (\neg\mathrm{B} \vee \mathrm{C})$.

Solution:

We consider a model $\mathcal{I}$ of KB. From $\mathcal{I} \models \mathrm{KB}$ follows $\mathcal{I} \models (\mathrm{A} \vee \neg\mathrm{B})$. From this follows that $\mathcal{I} \models \mathrm{A}$ or $\mathcal{I} \models \neg\mathrm{B}$.

In case $\mathcal{I} \models \mathrm{A}$ we can conclude from $\mathcal{I} \models \mathrm{KB}$ that $\mathcal{I} \models (\neg\mathrm{A} \vee \mathrm{C})$, and from this follows that $\mathcal{I} \models \neg\mathrm{A}$ or $\mathcal{I} \models \mathrm{C}$. The first case is in contradiction to the case we currently consider ($\mathcal{I} \models \mathrm{A}$), thus $\mathcal{I} \models \mathrm{C}$ must hold. From this we conclude that $\mathcal{I} \models (\neg\mathrm{B} \vee \mathrm{C})$.

In case $\mathcal{I} \models \neg\mathrm{B}$ we can directly conclude that $\mathcal{I} \models (\neg\mathrm{B} \vee \mathrm{C})$.