**Theory of Computer Science**

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# Exercise Sheet 1 — Solutions

**Exercise 1.1** (2 marks)

Prove with a direct proof: for all finite sets $S$, the power set $\mathcal{P}(S)$ has cardinality $2^{|S|}$.

**Solution:**

Consider an arbitrary finite set $S$. Each subset of $S$ can be "constructed" by iterating over all elements $e \in S$ and either including $e$ to the subset or not. Each such sequence of decisions results in a different subset. Thus, $S$ has $\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{|S| \text{ times}} = 2^{|S|}$ subsets and $|\mathcal{P}(S)| = 2^{|S|}$.

**Exercise 1.2** (2 marks)

Prove by contradiction that for all $n \in \mathbb{N}_0$ the following holds: if $n + 7$ is prime, then $n$ is not prime.

**Solution:**

Assume there is a number $n \in \mathbb{N}_0$, such that $n + 7$ and $n$ are prime.
Either $n$ or $n + 7$ have to be even since an even number increased by 7 is odd and an odd number increased by 7 is even. There is only one even prime (2) and $n + 7$ is definitvely larger. Thus, $n = 2$ must be true. But then $n + 7 = 9 = 3 \cdot 3$ is not prime. $\rightsquigarrow$ contradiction to the assumption that $n + 7$ and $n$ are prime.

**Exercise 1.3** (1 + 2 marks)

(a) Prove by mathematical induction that $n! > 2^n$ for all $n \geq 4$.

**Solution:**

*Induction basis* $n = 4$: $4! = 24 > 16 = 2^4$

*Induction hypothesis:* $k! > 2^k$ for all $4 \leq k \leq n$

*Inductive step:* $n \to n + 1$

$$\begin{aligned}(n+1)! &= (n+1) \cdot n! \\ &\overset{\text{IH}}{>} (n+1) \cdot 2^n \\ &> 2 \cdot 2^n = 2^{n+1}\end{aligned}$$

(b) Prove by induction over the number $n$ of elements in $S$ that for every finite set $S$ the power set $\mathcal{P}(S)$ has cardinality $2^{|S|}$.

**Solution:**

*Induction basis* $n = 0$ resp. $S = \emptyset$: $|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1 = 2^0$.

*Induction hypothesis:* for all finite sets $S$ with $|S| \leq n$ it holds that $|\mathcal{P}(S)| = 2^{|S|}$.

*Inductive step* $n \to n + 1$:

Let $S$ be an arbitrary set with $n + 1$ elements and let $e$ be some arbitrary element of $S$. Consider $S' = S \setminus \{e\}$. Since $|S'| = n$ we can use the induction hypothesis and conclude that there are $2^{|S'|}$ subsets $T \subseteq S'$. For every $T \subseteq S'$, $T$ itself and $T \cup \{e\}$ are subsets of $S$. These are all subsets of $S$ and they are all different. Hence, $|\mathcal{P}(S)| = 2|\mathcal{P}(S')| = 2 \cdot 2^n = 2^{n+1} = 2^{|S|}$.

**Exercise 1.4** (3 marks)

We inductively define a set of simple mathematical expressions which only utilize the following symbols: "Z", "T", "$\oplus$", "$\otimes$", "$[\![$", and "$]\!]$". The set $\mathcal{E}$ of *simple expressions* is inductively defined as follows:

- Z and T are simple expressions.
- If $x$ and $y$ are simple expressions, $[\![x \otimes y]\!]$ is also a simple expression.
- If $x$ and $y$ are simple expressions, $[\![x \oplus y]\!]$ is also a simple expression.

Examples for simple expressions: T, $[\![\mathtt{T} \otimes \mathtt{Z}]\!]$, $[\![[\![\mathtt{T} \otimes \mathtt{T}]\!] \oplus [\![\mathtt{Z} \oplus \mathtt{T}]\!]]\!]$

Furthermore we define a function $f : \mathcal{E} \to \mathbb{N}_0$ as follows:

- $f(\mathtt{Z}) = 0$, $f(\mathtt{T}) = 2$
- $f([\![x \otimes y]\!]) = f(x) \cdot f(y)$
- $f([\![x \oplus y]\!]) = f(x) + f(y)$

So for example: $f(\mathtt{T}) = 2$, $f([\![\mathtt{T} \otimes \mathtt{Z}]\!]) = f(\mathtt{T}) \cdot f(\mathtt{Z}) = 2 \cdot 0 = 0$, $f([\![[\![\mathtt{T} \otimes \mathtt{T}]\!] \oplus [\![\mathtt{Z} \oplus \mathtt{T}]\!]]\!]) = 6$.

Prove the following property for all simple expressions $x \in \mathcal{E}$ by structural induction:

$$f(x) \text{ is even.}$$

**Solution:**

We prove the statement by induction over the structure of simple expressions.

*Induction basis:* the statement is obviously true for all base cases, since $f(\mathtt{Z}) = 0$ and $f(\mathtt{T}) = 2$ are even.

*Induction hypothesis:* if $x$ and $y$ are partial expressions of a composite expression $z$, then $f(x)$ and $f(y)$ are even.

*Inductive step:* We have to show that the statement is true for composite expressions $z$, using the induction hypothesis that it is true for all partial expressions.

In case $z = [\![x \otimes y]\!]$ we have $f(z) = f(x) \cdot f(y)$. According to the induction hypothesis $f(x)$ and $f(y)$ are even, i.e., there are numbers $n, m \in \mathbb{Z}$ with $f(x) = 2n$ and $f(y) = 2m$. Then $f(z) = 2n \cdot 2m = 4nm$ is also even.

Analogously, in case $z = [\![x \oplus y]\!]$ we have $f(z) = f(x) + f(y)$. According to the induction hypothesis $f(x)$ and $f(y)$ are even, i.e., there are numbers $n, m \in \mathbb{Z}$ with $f(x) = 2n$ and $f(y) = 2m$. Then $f(z) = 2n + 2m = 2(n + m)$ is also even.