Theory of Computer Science

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Exercise Sheet 1

Due: Wednesday, February 27, 2019

Note: The goal of this exercise is to learn how to correctly express formal proofs. A formally correct proof consists of single steps where each step follows *immediately* from the previous steps or from the assumptions (for example when replacing a value by its definition). Please write down your proofs in detail and in a formal fashion. Examples can be found in the lecture slides.

Exercise 1.1 (2 marks)

Prove with a direct proof: for all finite sets S, the power set $\mathcal{P}(S)$ has cardinality $2^{|S|}$.

Exercise 1.2 (2 marks)

Prove by contradiction that for all $n \in \mathbb{N}_0$ the following holds: if $n + 7$ is prime, then n is not prime.

Hint: 2 is the only even prime number.

Exercise 1.3 (1 + 2 marks)

(a) Prove by mathematical induction that $n! > 2^n$ for all $n \geq 4$.

(b) Prove by induction over the number n of elements in S that for every finite set S the power set $\mathcal{P}(S)$ has cardinality $2^{|S|}$.

Exercise 1.4 (3 marks)

We inductively define a set of simple mathematical expressions which only utilize the following symbols: "Z", "T", "$\oplus$", "$\otimes$", "$[\![$", and "$]\!]$". The set $\mathcal{E}$ of *simple expressions* is inductively defined as follows:

- Z and T are simple expressions.
- If x and y are simple expressions, $[\![x \otimes y]\!]$ is also a simple expression.
- If x and y are simple expressions, $[\![x \oplus y]\!]$ is also a simple expression.

Examples for simple expressions: T, $[\![\mathtt{T} \otimes \mathtt{Z}]\!]$, $[\![[\![\mathtt{T} \otimes \mathtt{T}]\!] \oplus [\![\mathtt{Z} \oplus \mathtt{T}]\!]]\!]$

Furthermore we define a function $f : \mathcal{E} \to \mathbb{N}_0$ as follows:

- $f(\mathtt{Z}) = 0$, $f(\mathtt{T}) = 2$
- $f([\![x \otimes y]\!]) = f(x) \cdot f(y)$
- $f([\![x \oplus y]\!]) = f(x) + f(y)$

So for example: $f(\mathtt{T}) = 2$, $f([\![\mathtt{T} \otimes \mathtt{Z}]\!]) = f(\mathtt{T}) \cdot f(\mathtt{Z}) = 2 \cdot 0 = 0$, $f([\![[\![\mathtt{T} \otimes \mathtt{T}]\!] \oplus [\![\mathtt{Z} \oplus \mathtt{T}]\!]]\!]) = 6$.

Prove the following property for all simple expressions $x \in \mathcal{E}$ by structural induction:

$$f(x) \text{ is even.}$$