Theory of Computer Science

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Exercise meeting 9 — Solutions

Exercise 9.1

Consider propositional formula $\varphi = \neg(A \lor (\neg B \land C))$.

(a) Specify formula χ_{all} as it is used in the polynomial reduction of SAT to 3SAT.

Solution:

Formula χ_{all} over $\{X_A, X_B, X_C, X_{\neg B}, X_{(\neg B \land C)}, X_{(A \lor (\neg B \land C))}, X_{\neg(A \lor (\neg B \land C))}\}$ is built from the following subformulas:

$$\begin{aligned}
\chi_A &= (X_A \leftrightarrow A)\\
\chi_B &= (X_B \leftrightarrow B)\\
\chi_C &= (X_C \leftrightarrow C)\\
\chi_{\neg B} &= (X_{\neg B} \leftrightarrow \neg X_B)\\
\chi_{(\neg B \land C)} &= (X_{(\neg B \land C)} \leftrightarrow (X_{\neg B} \land X_C))\\
\chi_{(A \lor (\neg B \land C))} &= (X_{(A \lor (\neg B \land C))} \leftrightarrow (X_A \lor X_{(\neg B \land C)}))\\
\chi_{\neg(A \lor (\neg B \land C))} &= (X_{\neg(A \lor (\neg B \land C))} \leftrightarrow \neg X_{(A \lor (\neg B \land C))})
\end{aligned}$$

Then

$$\chi_{\text{all}} = \chi_A \land \chi_B \land \chi_B \land \chi_{\neg B} \land \chi_{(\neg B \land C)} \land \chi_{(A \lor (\neg B \land C))} \land \chi_{\neg(A \lor (\neg B \land C))},$$

always using the logically equivalent CNF formula that results from replacing the abbreviation $\leftrightarrow$ with the corresponding formula, e.g. $\chi_{(A \lor (\neg B \land C))} \equiv (X_{(A \lor (\neg B \land C))} \rightarrow (X_A \lor X_{(\neg B \land C)})) \land ((X_A \lor X_{(\neg B \land C)}) \rightarrow X_{(A \lor (\neg B \land C))}) \equiv (\neg X_{(A \lor (\neg B \land C))} \lor (X_A \lor X_{(\neg B \land C)})) \land (\neg(X_A \lor X_{(\neg B \land C)}) \lor X_{(A \lor (\neg B \land C))})$.

(b) $\mathcal{I} = \{A \mapsto F, B \mapsto T, C \mapsto T\}$ is a model of φ. Specify the corresponding model of χ_{all}.

Solution:

$$\begin{aligned}
\mathcal{I}' = \{&X_A \mapsto F, X_B \mapsto T, X_C \mapsto T, X_{\neg B} \mapsto F, X_{(\neg B \land C)} \mapsto F,\\
&X_{(A \lor (\neg B \land C))} \mapsto F, X_{\neg(A \lor (\neg B \land C))} \mapsto T\}
\end{aligned}$$

Exercise 9.2

The decision problem SAT(satisfiability) is defined as follows:

Given: a propositional logic formula φ

Question: Is φ satisfiable?

The general problem GenSAT(model generation) is defined as follows:

Given: a propositional logic formula φ

Output: a model for φ or a message that none exists

Show that if there is a polynomial algorithm for SAT then there is a polynomial algorithm for GenSAT.

Solution:

We specify an algorithm that solves GenSAT. It uses transformations $\psi[v \mapsto T]$ and $\psi[v \mapsto F]$ that replace every occurrence of v with a small valid (e.g. $v \vee \neg v$) and unsatisfiable (e.g. $v \wedge \neg v$) formula, respectively. Then

- $\mathcal{I}'$ is a model of $\psi[v \mapsto T]$ iff $\mathcal{I}$ with $\mathcal{I}(v) = T$ and $\mathcal{I}(v') = \mathcal{I}'(v')$ for $v' \neq v$ is a model of ψ and
- $\mathcal{I}'$ is a model of $\psi[v \mapsto F]$ iff $\mathcal{I}$ with $\mathcal{I}(v) = F$ and $\mathcal{I}(v') = \mathcal{I}'(v')$ for $v' \neq v$

Moreover, if ψ is satisfiable and $\psi[v \mapsto T]$ is unsatisfiable then $\psi[v \mapsto F]$ is satisfiable.

The algorithm proceeds as follows:

Call the algorithm for SAT on input φ. If it is unsatisfiable, output that it has no model.

Otherwise, we successively build a model $\mathcal{I}$ for φ, starting with $\varphi' := \varphi$, as follows:

While there is still an unassigned variable v, call the SAT algorithm for $\varphi'[v \mapsto T]$. If the answer is yes, set $\mathcal{I}(v) = T$ and continue with $\varphi' := \varphi'[v \mapsto T]$, otherwise set $\mathcal{I}(v) = F$ and continue with $\varphi' := \varphi'[v \mapsto F]$.

Since the number of variables is bound by the size of φ, there is only a polynomial number of iterations. The size of the last formula φ' is at most k times larger than φ for some constant k ($k = 2$ if we only count variable occurrences to determine the size). Hence, if every call to SAT is possible in polynomial time then the overall runtime is polynomial.