Theory of Computer Science

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Exercise meeting 8 — Solutions

Exercise 8.1

This exercise was a question in the exam in 2017.

Consider the following decision problems:

DirHamiltonPath:

- *Given:* directed graph $G = \langle V, E \rangle$
- *Question:* Does G contain a Hamilton path?

DirHamiltonPathWithStartPoint:

- *Given:* directed graph $G = \langle V, E \rangle$, start vertex $v_\text{s} \in V$
- *Question:* Does G contain a Hamilton path with start vertex v_s, i.e., a Hamilton path $\pi = \langle v_1, \dots, v_n \rangle$ with $v_1 = v_\text{s}$?

(a) Show DirHamiltonPathWithStartPoint $\in$ NP by specifying a non-deterministic polynomial algorithm.

Solution:

A possible algorithm will proceed iteratively by guessing nodes one after the other, without repetitions, starting from the given starting vertex v_s and making sure that they conform a path, i.e. that an edge exists between each node and the following one.

$current := v_\text{s}$
$remaining := V \setminus \{v_\text{s}\}$
WHILE $remaining \neq \emptyset$:
  GUESS $next \in remaining$
  IF $\langle current, next \rangle \notin E$ THEN REJECT
  $remaining := remaining \setminus \{next\}$
  $current := next$
ACCEPT

(b) Prove that DirHamiltonPathWithStartPoint is NP-hard. You may use that DirHamiltonPath is NP-complete.

Solution:

A possible way of proving NP-hardness is to show that

$$\text{DirHamiltonPath} \leq_\text{p} \text{DirHamiltonPathWithStartPoint}$$

Intuitively, if we have a decision procedure for DirHamiltonPathWithStartPoint, then we can decide whether a given graph $G = \langle V, E \rangle$ contains a Hamilton path as follows: we construct a graph G' which is like G, but contains an *extra* vertex u and extra edges from u to all the rest of vertices. G will have a Hamilton path iff the new graph G' contains a Hamilton path starting at u, and the construction of G' can be done in polynomial time.

Formally:

$$f(\langle V, E\rangle) = \langle G', u\rangle,$$

where u is a *new* node (i.e. $u \notin V$) and G' is the graph with vertices $V' = V \cup \{u\}$ and edges $E' = E \cup \{\langle u, v\rangle \mid v \in V\}$. The function f is total and can be computed in polynomial time (it only adds one vertex and $|V|$ edges).

We need to prove that $G = \langle V, E\rangle$ has a Hamilton path if and only if $f(G) = \langle G', u\rangle$ has a Hamilton path starting at u.

Forward implication: Assume $\pi = \langle v_1, \ldots, v_n\rangle$ is a Hamilton path in G, and consider the sequence $\pi' = \langle u, v_1, \ldots, v_n\rangle$ of vertices from G'. π' is a path, since G' contains an edge between u and v_1 by construction, and it contains an edge between each pair $\langle v_i, v_{i+1}\rangle$, $1 \leq i < n$, simply because we know that π is a path in G, and G' contains all edges from G. Additionally, π' contains all $n + 1$ vertices from V' *without repetition*, as π contains all vertices in V without repetition, and we know that $u \notin V$ and that $V' = V \cup \{u\}$. We thus conclude that π' is a Hamilton path in G' starting at vertex u.

Backward implication: Assume now that $\pi' = \langle u, v_1, \ldots, v_n\rangle$ is a Hamilton path in G' starting at u, and consider the sequence $\pi = \langle v_1, \ldots, v_n\rangle$ of vertices from G. π is a path in G, since all edges in E' that affect nodes from $\{v_1, \ldots, v_n\}$ are also contained in E, and we know π' is a path in G'. Additionally, π contains all vertices from V exactly once, since we know that π' contains no repetition (and thus no subsequence of it can contain a repetition), and $V = V' \setminus \{u\}$. Hence, π is a Hamilton path in G.

Reminder: A *Hamilton path* in a graph $\langle V, E\rangle$ is a vertex sequence $\pi = \langle v_1, \ldots, v_n\rangle$ that defines a path ($\langle v_i, v_{i+1}\rangle \in E$ for all $1 \leq i < n$) and includes every graph vertex exactly once.