Theory of Computer Science

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Spring Term 2019

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Exercise meeting 2 — Solutions

Exercise 2.1 (Formula Sets and Resolution)
Use the resolution calculus to show for the set of formulas

$$\text{KB} = \{(A \lor B), (\neg B \lor C)\}$$

that $\text{KB} \models (A \lor C)$.

Solution:

It holds that $\text{KB} \models (A \lor C)$ iff $\text{KB} \cup \{\neg(A \lor C)\}$ is unsatisfiable.
$\text{KB} \cup \{\neg(A \lor C)\}$ corresponds to the set of clauses $\Delta = \{\{A, B\}, \{\neg B, C\}, \{\neg A\}, \{\neg C\}\}$.
We can derive $\square$ as follows:

$$\begin{aligned}
C_1 &= \{A, B\} \quad (\text{from } \Delta)\\
C_2 &= \{\neg B, C\} \quad (\text{from } \Delta)\\
C_3 &= \{\neg A\} \quad (\text{from } \Delta)\\
C_4 &= \{\neg C\} \quad (\text{from } \Delta)\\
C_5 &= \{B\} \quad (\text{from } C_1, C_3)\\
C_6 &= \{C\} \quad (\text{from } C_2, C_5)\\
C_7 &= \square \quad (\text{from } C_4, C_6)
\end{aligned}$$

Hence, Δ is unsatisfiable.

Exercise 2.2 (Syntax of Predicate Logic)
Which of the following expressions are syntactically correct formulas or terms for the following signature $\mathcal{S}$? Analyse also all subformulas and all subterms. For the formulas, also state what kind of formulas they are (atomic formula, conjunction, ...).

$$\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$$

with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\text{c}\}$, $\mathcal{F} = \{\text{f}, \text{g}, \text{h}\}$, where $ar(f) = 3$, $ar(g) = ar(h) = 1$, and $\mathcal{P} = \{\text{Q}, \text{R}, \text{S}\}$, where $ar(Q) = 2$, $ar(R) = ar(S) = 1$.

(a) $\text{f}(x, y, z)$

Solution:

Term $\text{f}(x, y, z)$ with subterms x, y and z.

(b) $\text{f}(x, y)$

Solution:

Syntactically not correct because function symbol f has arity 3.

(c) $\mathrm{Q}(x, y)$

Solution:

Atomic formula $\mathrm{Q}(x, y)$ with subterms x and y.

(d) $(\mathrm{g}(x) = \mathrm{R}(y))$

Solution:

$(\mathrm{g}(x) = \mathrm{R}(y))$ is syntactically not correct, because $\mathrm{R}(y)$ is a formula but the identity $(=)$ would require a term.

(e) $(\mathrm{g}(x) = \mathrm{f}(y, \mathrm{c}, \mathrm{h}(x)))$

Solution:

The formula $(\mathrm{g}(x) = \mathrm{f}(y, \mathrm{c}, \mathrm{h}(x)))$ is an identity with sub-terms $\mathrm{g}(x)$ (with subterm x) and $\mathrm{f}(y, \mathrm{c}, \mathrm{h}(x))$. The latter has subterms x (a variable), c (a constant) and $\mathrm{h}(x)$ (a function term with variable x as subterm).

(f) $\forall \mathrm{c}\, \mathrm{Q}(c, x)$

Solution:

Syntactically not correct because the quantifier ranges over a constant, not a variable.

(g) $(\mathrm{R}(x) \wedge \forall x\, \mathrm{S}(x))$

Solution:

Formula $(\mathrm{R}(x) \wedge \forall x\, \mathrm{S}(x))$ is a conjunction with atomic sub-formula R(x) (with sub-term x) and universal quantification $\forall x\, \mathrm{S}(x)$. The latter formula has an (atomic) sub-formula $\mathrm{S}(x)$ with term x.

(h) $(\mathrm{g}(\mathrm{h}(x)) \wedge R(x))$

Solution:

Syntactically not correct because a conjunction is build from formulas but $\mathrm{g}(\mathrm{h}(x))$ is a term.

(i) $(\forall x \exists y\, (\mathrm{g}(x) = y) \vee (\mathrm{h}(x) = \mathrm{c}))$

Solution:

This is a disjunction with subformulas $\forall x \exists y\, (\mathrm{g}(x) = y)$ and $(\mathrm{h}(x) = \mathrm{c})$.

The universal quantification $\forall x \exists y\, (\mathrm{g}(x) = y))$ has subformula $\exists y\, (\mathrm{g}(x) = y)$, which is an existential quantification with (identity) subformula $(\mathrm{g}(x) = y)$ build over terms $\mathrm{g}(x)$ (with sub-term x) and y.

Formula $(\mathrm{h}(x) = \mathrm{c})$ is an identity with sub-terms $\mathrm{h}(x)$ and c.