Theory of Computer Science

G. Röger | University of Basel
Spring Term 2019 | Computer Science

Exercise meeting 1 — Solutions

Exercise 1.1 (Expressing Formulas in Propositional Logic)

Write down the following statements as propositional logic formulas. In order to do so, also define appropriate atomic propositions.

(a) "If the traffic light is red, then the car may not drive."

(b) "The car may drive if and only if the traffic light is not red and there is no pedestrian on the street."

Solution:

(a) $(\mathrm{TrafficLightRed} \rightarrow \neg\mathrm{CarMayDrive})$

(b) $(\mathrm{CarMayDrive} \leftrightarrow (\neg\mathrm{TrafficLightRed} \land \neg\mathrm{Pedestrian}))$

Exercise 1.2 (Truth tables)

Let $A = \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ be a set of propositional variables and $\varphi = ((\mathrm{X} \land \mathrm{Y}) \rightarrow \mathrm{Z})$ be a propositional formula over A. Specify the truth table for φ.

Use the truth table to decide whether φ is satisfiable, unsatisfiable, valid and/or falsifiable.

Solution:

$\mathcal{I}(\mathrm{X})$	$\mathcal{I}(\mathrm{Y})$	$\mathcal{I}(\mathrm{Z})$	$\mathcal{I} \models (\mathrm{X} \land \mathrm{Y})$	$\mathcal{I} \models ((\mathrm{X} \land \mathrm{Y}) \rightarrow \mathrm{Z})$
0	0	0	No	Yes
0	0	1	No	Yes
0	1	0	No	Yes
0	1	1	No	Yes
1	0	0	No	Yes
1	0	1	No	Yes
1	1	0	Yes	No
1	1	1	Yes	Yes

As $\{\mathrm{X} \mapsto 0, \mathrm{Y} \mapsto 0, \mathrm{Z} \mapsto 0\} \models \varphi$ the formula is satisfiable and not unsatisfiable.
Due to $\{\mathrm{X} \mapsto 1, \mathrm{Y} \mapsto 1, \mathrm{Z} \mapsto 0\} \not\models \varphi$ it is falisfiable and hence not valid.

Exercise 1.3 (Semantics of Propositional Logic)

Let $\varphi = ((\mathrm{X} \land \mathrm{Y}) \lor \neg\mathrm{X})$ be a propositional formula over $\{\mathrm{X}, \mathrm{Y}\}$. Consider interpretation $\mathcal{I} = \{\mathrm{X} \mapsto 1, \mathrm{Y} \mapsto 1\}$ for $\{\mathrm{X}, \mathrm{Y}\}$ and show by applying the semantics of propositional logic that $\mathcal{I}$ is a model of φ (i.e. $\mathcal{I} \models \varphi$).

Solution:

Since $\mathcal{I}(\mathrm{X}) = 1$, it holds that $\mathcal{I} \models \mathrm{X}$. Analogously, we know from $\mathcal{I}(\mathrm{Y}) = 1$ that $\mathcal{I} \models \mathrm{Y}$. Using the semantics of the conjunction $\land$, we conclude that $\mathcal{I} \models (\mathrm{X} \land \mathrm{Y})$. With the semantics of the disjunction $\lor$, we get that $\mathcal{I} \models ((\mathrm{X} \land \mathrm{Y}) \lor \neg\mathrm{X}) = \varphi$ (this is independent on whether $\mathcal{I} \models \neg\mathrm{X}$ or not, so it is unnecessary to show that indeed $\mathcal{I} \not\models \neg\mathrm{X}$).

Exercise 1.4 (Properties of Propositional Logic Formulas)

Show *without* a truth table that $\varphi = (\mathrm{A} \rightarrow (\mathrm{B} \leftrightarrow \mathrm{C}))$ is falsifiable. Is φ valid?

Solution:

We consider the following interpretation $\mathcal{I} = \{\mathrm{A} \mapsto 1, \mathrm{B} \mapsto 1, \mathrm{C} \mapsto 0\}$.

Since $\mathcal{I}(\mathrm{C}) = 0$, $\mathcal{I} \not\models \mathrm{C}$ holds. Since $\mathcal{I}(\mathrm{B}) = 1$, $\mathcal{I} \models \mathrm{B}$ holds and thus $\mathcal{I} \not\models \neg\mathrm{B}$ holds as well.
Together with the semantics for disjunction the above statements result in $\mathcal{I} \not\models (\neg\mathrm{B} \vee \mathrm{C})$. By using the semantics for conjunction it follows that $\mathcal{I} \not\models ((\neg\mathrm{B} \vee \mathrm{C}) \wedge (\neg\mathrm{C} \vee \mathrm{B}))$. This can be abbreviated by $\mathcal{I} \not\models ((\mathrm{B} \rightarrow \mathrm{C}) \wedge (\mathrm{C} \rightarrow \mathrm{B}))$, or even shorter by $\mathcal{I} \not\models (\mathrm{B} \leftrightarrow \mathrm{C})$ (1).
Since $\mathcal{I}(\mathrm{A}) = 1$, $\mathcal{I} \models \mathrm{A}$ holds. Using the semantics of negation it follows that $\mathcal{I} \not\models \neg\mathrm{A}$ (2). Using the semantics for disjunction we can conclude from (1) and (2) that $\mathcal{I} \not\models (\neg\mathrm{A} \vee (\mathrm{B} \leftrightarrow \mathrm{C}))$, which can be abbreviated by $\mathcal{I} \not\models (\mathrm{A} \rightarrow (\mathrm{B} \leftrightarrow \mathrm{C}))$.
From this we can conclude that interpretation $\mathcal{I}$ does not satisfy φ. Thus φ is falsifiable and also not valid.