# Foundations of Artificial Intelligence

## 22. Constraint Satisfaction Problems: Introduction and Examples

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# Classification

Classification:

## Constraint Satisfaction Problems

environment:

- static vs. dynamic
- deterministic vs. non-deterministic vs. stochastic
- fully vs. partially vs. not observable
- discrete vs. continuous
- single-agent vs. multi-agent

problem solving method:

- problem-specific vs. general vs. learning

# Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- 22.–23. Introduction
  - 22. Introduction and Examples
  - 23. Constraint Networks
- 24.–26. Basic Algorithms
- 27.–28. Problem Structure

# Introduction

# Constraints

## What is a Constraint?

a condition that every solution to a problem must satisfy

German: Einschränkung, Nebenbedingung (math.)

Examples: Where do constraints occur?

- mathematics: requirements on solutions of optimization problems (e.g., equations, inequalities)
- software testing: specification of invariants to check data consistency (e.g., assertions)
- databases: integrity constraints

# Constraint Satisfaction Problems: Informally

Given:

- set of variables with corresponding domains
- set of constraints that the variables must satisfy
  - most commonly binary, i.e., every constraint refers to two variables

Solution:

- assignment to the variables that satisfies all constraints

German: Variablen, Constraints, binär, Belegung

# Examples

# Examples

## Examples

- 8 queens problem
- Latin squares
- Sudoku
- graph coloring
- satisfiability in propositional logic

German: 8-Damen-Problem, lateinische Quadrate, Sudoku, Graphfärbung, Erfüllbarkeitsproblem der Aussagenlogik

more complex examples:

- systems of equations and inequalities
- database queries

# Example: 8 Queens Problem (Reminder)

(reminder from previous two chapters)

## 8 Queens Problem

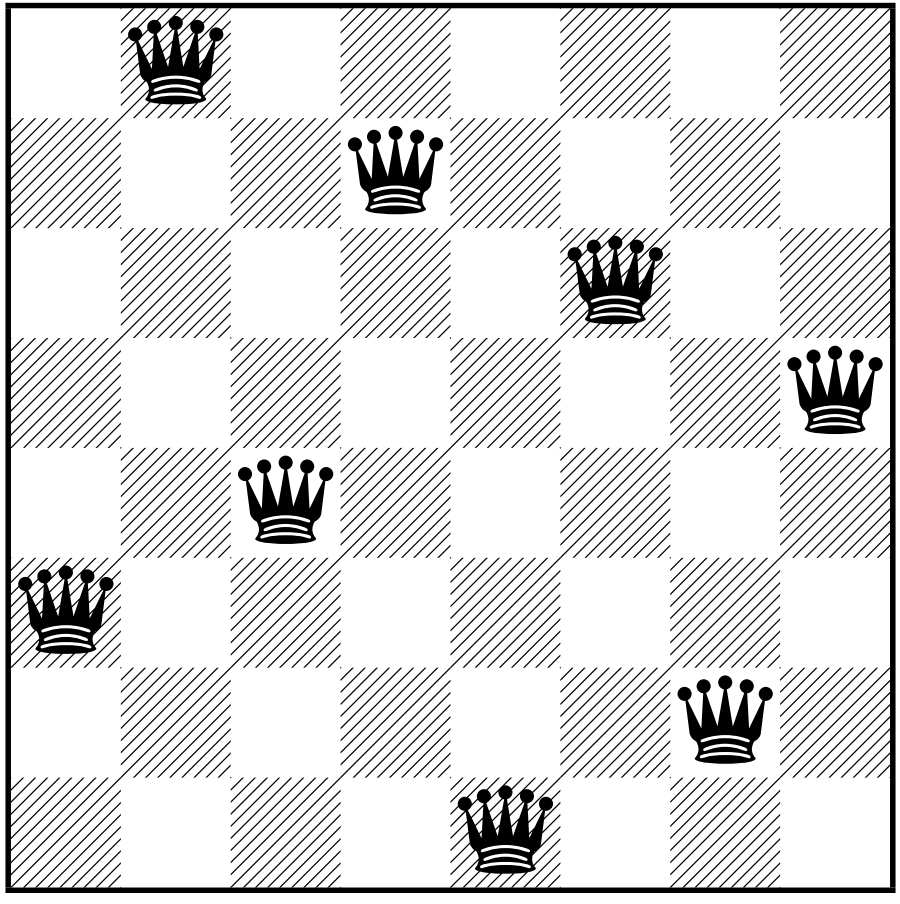
How can we

- place 8 queens on a chess board
- such that no two queens threaten each other?

- originally proposed in 1848
- variants: board size; other pieces; higher dimension

There are 92 solutions, or 12 solutions if we do not count symmetric solutions (under rotation or reflection) as distinct.

# 8 Queens Problem: Example Solution



example solution for the 8 queens problem

# Example: Latin Squares

## Latin Squares

How can we

- build an $n \times n$ matrix
  with $n$ symbols
- such that every symbol occurs exactly once
  in every row and every column?

$$[1] \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

There exist 12 different Latin squares of size 3,
576 of size 4, 161 280 of size 5, . . . ,
5 524 751 496 156 892 842 531 225 600 of size 9.

# Example: Sudoku

## Sudoku

How can we

- completely fill an already partially filled $9 \times 9$ matrix with numbers between 1–9
- such that each row, each column, and each of the nine $3 \times 3$ blocks contains every number exactly once?

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 5 | | | 3 | | 9 | | 1 |
| | 1 | | | | 4 | | | |
| 4 | | 7 | | | | 2 | | 8 |
| | | 5 | 2 | | | | | |
| | | | | 9 | 8 | 1 | | |
| | 4 | | | | 3 | | | |
| | | | 3 | 6 | | | 7 | 2 |
| | 7 | | | | | | | 3 |
| 9 | | 3 | | | | 6 | | 4 |

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| 6 | 1 | 9 | 8 | 2 | 4 | 3 | 5 | 7 |
| 4 | 3 | 7 | 9 | 1 | 5 | 2 | 6 | 8 |
| 3 | 9 | 5 | 2 | 7 | 1 | 4 | 8 | 6 |
| 7 | 6 | 2 | 4 | 9 | 8 | 1 | 3 | 5 |
| 8 | 4 | 1 | 6 | 5 | 3 | 7 | 2 | 9 |
| 1 | 8 | 4 | 3 | 6 | 9 | 5 | 7 | 2 |
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| 3 | 9 | 5 | 2 | 7 | 1 | 4 | 8 | 6 |
| 7 | 6 | 2 | 4 | 9 | 8 | 1 | 3 | 5 |
| 8 | 4 | 1 | 6 | 5 | 3 | 7 | 2 | 9 |
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| 5 | 7 | 6 | 1 | 4 | 2 | 8 | 9 | 3 |
| 9 | 2 | 3 | 5 | 8 | 7 | 6 | 1 | 4 |

relationship to Latin squares?

# Sudoku: Trivia

- well-formed Sudokus have exactly one solution
- to achieve well-formedness, $\geq 17$ cells must be filled already (McGuire et al., 2012)
- 6 670 903 752 021 072 936 960 solutions
- only 5 472 730 538 “non-symmetrical” solutions

# Example: Graph Coloring

## Graph Coloring

How can we

- color the vertices of a given graph using $k$ colors
- such that two neighboring vertices never have the same color?

(The graph and $k$ are problem parameters.)

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# Four Color Problem

famous problem in mathematics: Four Color Problem

- Is it always possible to color a planar graph with 4 colors?
- conjectured by Francis Guthrie (1852)
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- 1890 first proof that 5 colors suffice
- several wrong proofs surviving for over 10 years
- solved by Appel and Haken in 1976: 4 colors suffice
- Appel and Haken reduced the problem to 1936 cases, which were then checked by computers
- first famous mathematical problem solved (partially) by computers
  $\rightsquigarrow$ led to controversy: is this a mathematical proof?

Numberphile video:

`https://www.youtube.com/watch?v=NgbK43jB4rQ`

# Satisfiability in Propositional Logic

## Satisfiability in Propositional Logic

How can we

- assign truth values (true/false) to a set of propositional variables
- such that a given set of clauses (formulas of the form $X \vee \neg Y \vee Z$) is satisfied (true)?

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remarks:

- NP-complete (Cook 1971; Levin 1973)
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relationship to previous problems (e.g., Sudoku)?

# Practical Applications

- There are thousands of practical applications of constraint satisfaction problems.
- This statement is true already for the satisfiability problem of propositional logic.

some examples:

- verification of hardware and software
- timetabling (e.g., generating time schedules, room assignments for university courses)
- assignment of frequency spectra (e.g., broadcasting, mobile phones)

# Summary

# Summary

- constraint satisfaction:
  - find assignment for a set of variables
  - with given variable domains
  - that satisfies a given set of constraints.
- examples:
  - 8 queens problem
  - Latin squares
  - Sudoku
  - graph coloring
  - satisfiability in propositional logic
  - many practical applications