Planning and Optimization

B5. Computational Complexity of Planning: Background

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October 18, 2017

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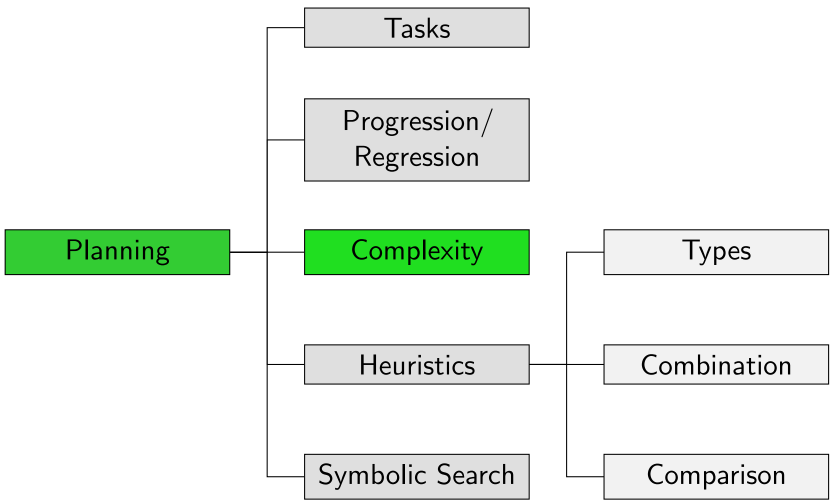
Content of this Course



B5.1 Motivation

How Difficult is Planning?

- Using progression and a state-space search algorithm like breadth-first search, planning can be solved in polynomial time in the size of the transition system (i.e., the number of states).
- However, the number of states is exponential in the number of state variables, and hence in general exponential in the size of the input to the planning algorithm.

⇝ Do non-exponential planning algorithms exist?

⇝ What is the precise computational complexity of planning?

Why Computational Complexity?

- understand the problem
- know what is not possible
- find interesting subproblems that are easier to solve
- distinguish essential features from syntactic sugar
  - Is STRIPS planning easier than general planning?
  - Is planning for FDR tasks harder than for propositional tasks?

B5.2 Background: Turing Machines

Reminder: Complexity Theory

Need to Catch Up?

- This and the following section are mostly reminders.
- We assume knowledge of complexity theory:
  - languages and decision problems
  - Turing machines: NTMs and DTMs; polynomial equivalence with other models of computation
  - complexity classes: P and NP
  - polynomial reductions
- If you are not familiar with these topics, we recommend Parts C and E of the Theorie der Informatik course at `http://cs.unibas.ch/fs2017/theorie-der-informatik/`.
- The slides are in English, even though the course is not.

Note: the space complexity classes (DSPACE, NSPACE, PSPACE, NPSPACE) go beyond the content of the prerequisite course.

Nondeterministic Turing Machines

Definition (Nondeterministic Turing Machine)

A nondeterministic Turing machine (NTM) is a 6-tuple $\langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ with the following components:

- input alphabet Σ and blank symbol $\square \notin \Sigma$
  - alphabets always nonempty and finite
  - tape alphabet $\Sigma_\square = \Sigma \cup \{\square\}$
- finite set Q of internal states with initial state $q_0 \in Q$ and accepting state $q_Y \in Q$
  - nonterminal states $Q' := Q \setminus \{q_Y\}$
- transition relation $\delta \subseteq (Q' \times \Sigma_\square) \times (Q \times \Sigma_\square \times \{-1, +1\})$

Deterministic Turing Machines

Definition (Deterministic Turing Machine)

A deterministic Turing machine (DTM) is an NTM where the transition relation is functional, i.e., for all $\langle q, a \rangle \in Q' \times \Sigma_\square$, there is exactly one triple $\langle q', a', \Delta \rangle$ with $\langle \langle q, a \rangle, \langle q', a', \Delta \rangle \rangle \in \delta$.

Notation: We write $\delta(q, a)$ for the unique triple $\langle q', a', \Delta \rangle$ such that $\langle \langle q, a \rangle, \langle q', a', \Delta \rangle \rangle \in \delta$.

Turing Machine Configurations

Definition (Configuration)

Let $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ be an NTM.

A configuration of M is a triple $\langle w, q, x \rangle \in \Sigma_\square^* \times Q \times \Sigma_\square^+$.

- w: tape contents to the left of tape head
- q: current state
- x: tape contents at tape head and to its right

Turing Machine Transitions

Definition (Yields)

Let $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ be an NTM.

A configuration c of M yields a configuration c' of M, in symbols $c \vdash c'$, as defined by the following rules, where $a, a', b \in \Sigma_\square$, $w, x \in \Sigma_\square^*$, $q, q' \in Q$ and $\langle \langle q, a \rangle, \langle q', a', \Delta \rangle \rangle \in \delta$:

$$
\begin{aligned}
\langle w, q, ax \rangle &\vdash \langle wa', q', x \rangle && \text{if } \Delta = +1, |x| \geq 1 \\
\langle w, q, a \rangle &\vdash \langle wa', q', \square \rangle && \text{if } \Delta = +1 \\
\langle wb, q, ax \rangle &\vdash \langle w, q', ba'x \rangle && \text{if } \Delta = -1 \\
\langle \varepsilon, q, ax \rangle &\vdash \langle \varepsilon, q', \square a'x \rangle && \text{if } \Delta = -1
\end{aligned}
$$

Accepting Configurations

Definition (Accepting Configuration: Time)

Let $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ be an NTM,
let $c = \langle w, q, x \rangle$ be a configuration of M, and let $n \in \mathbb{N}_0$.

- If $q = q_Y$, M accepts c in time n.
- If $q \neq q_Y$ and M accepts some c' with $c \vdash c'$ in time n, then M accepts c in time $n + 1$.

Definition (Accepting Configuration: Space)

Let $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ be an NTM,
let $c = \langle w, q, x \rangle$ be a configuration of M, and let $n \in \mathbb{N}_0$.

- If $q = q_Y$ and $|w| + |x| \leq n$, M accepts c in space n.
- If $q \neq q_Y$ and M accepts some c' with $c \vdash c'$ in space n, then M accepts c in space n.

Note: "in time/space n" means at most n, not exactly n

Accepting Words and Languages

Definition (Accepting Words)

Let $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ be an NTM.

M accepts the word $w \in \Sigma^*$ in time (space) $n \in \mathbb{N}_0$
iff M accepts $\langle \varepsilon, q_0, w \rangle$ in time (space) n.

- Special case: M accepts ε in time (space) $n \in \mathbb{N}_0$ iff M accepts $\langle \varepsilon, q_0, \square \rangle$ in time (space) n.

Definition (Accepting Languages)

Let $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ be an NTM, and let $f : \mathbb{N}_0 \to \mathbb{N}_0$.

M accepts the language $L \subseteq \Sigma^*$ in time (space) f
iff M accepts each word $w \in L$ in time (space) $f(|w|)$,
and M does not accept any word $w \notin L$ (in any time/space).

B5.3 Background: Complexity Classes

Time and Space Complexity Classes

Definition (DTIME, NTIME, DSPACE, NSPACE)

Let $f : \mathbb{N}_0 \to \mathbb{N}_0$.

Complexity class DTIME(f) contains all languages accepted in time f by some DTM.

Complexity class NTIME(f) contains all languages accepted in time f by some NTM.

Complexity class DSPACE(f) contains all languages accepted in space f by some DTM.

Complexity class NSPACE(f) contains all languages accepted in space f by some NTM.

Polynomial Time and Space Classes

Let $\mathcal{P}$ be the set of polynomials $p : \mathbb{N}_0 \to \mathbb{N}_0$ whose coefficients are natural numbers.

Definition (P, NP, PSPACE, NPSPACE)

$$\begin{aligned} \mathrm{P} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathrm{DTIME}(p) \\ \mathrm{NP} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathrm{NTIME}(p) \\ \mathrm{PSPACE} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathrm{DSPACE}(p) \\ \mathrm{NPSPACE} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathrm{NSPACE}(p) \end{aligned}$$

Polynomial Complexity Class Relationships

Theorem (Complexity Class Hierarchy)

$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE}$

Proof.

$\mathrm{P} \subseteq \mathrm{NP}$ and $\mathrm{PSPACE} \subseteq \mathrm{NPSPACE}$ are obvious because deterministic Turing machines are a special case of nondeterministic ones.

$\mathrm{NP} \subseteq \mathrm{NPSPACE}$ holds because a Turing machine can only visit polynomially many tape cells within polynomial time.

$\mathrm{PSPACE} = \mathrm{NPSPACE}$ is a special case of a classical result known as Savitch's theorem (Savitch 1970). □

B5.4 Summary

Summary

- We recalled the definitions of the most important complexity classes from complexity theory:
  - P: decision problems solvable in polynomial time
  - NP: decision problems solvable in polynomial time by non-deterministic algorithms
  - PSPACE: decision problems solvable in polynomial space
  - NPSPACE: decision problems solvable in polynomial space by non-deterministic algorithms
- These classes are related by $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE}$.