Planning and Optimization

C23. Linear & Integer Programming

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Examples

Linear Program: Example Maximization Problem

Example

$$\text{maximize} \quad 2x - 3y + z \quad \text{subject to}$$

$$\begin{aligned} x + 2y + z &\leq 10 \\ x \qquad - z &\leq 0 \end{aligned}$$

$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

$\rightsquigarrow$ unique optimal solution:
$x = 5$, $y = 0$, $z = 5$ (objective value 15)

Example: Diet Problem

- n different types of food $F_1, \ldots, F_n$
- m different nutrients $N_1, \ldots, N_m$
- The minimum daily requirement of nutrient N_j is r_j.
- The amount of nutrient N_j in one unit of food F_i is a_{ij}.
- One unit of food F_i costs c_i.

How to supply the required nutrients at minimum cost?

Example: Diet Problem

- Use LP variable x_i for the number of units of food F_i purchased per day.
- The cost per day is $\sum_{i=1}^{n} c_i x_i$.
- The amount of nutrient N_j in this diet is $\sum_{i=1}^{n} a_{ij} x_i$.
- The minimum daily requirement for each nutrient N_j must be met: $\sum_{i=1}^{n} a_{ij} x_i \geq r_j$ for $1 \leq j \leq m$
- We can't buy negative amounts of food: $x_i \geq 0$ for $1 \leq i \leq n$
- We want to minimize the cost of food.

Diet Problem: Linear Program

Example (Linear Program for Diet Problem)

minimize $\sum_{i=1}^{n} c_i x_i$ subject to

$$\sum_{i=1}^{n} a_{ij} x_i \geq r_j \qquad \text{for } 1 \leq j \leq m$$

$$x_i \geq 0 \qquad \text{for } 1 \leq i \leq n$$

Linear Programs

Linear Programs and Integer Programs

Linear Program

A linear program (LP) consists of:

- a finite set of real-valued variables V
- a finite set of linear inequalities (constraints) over V
- an objective function, which is a linear combination of V
- which should be minimized or maximized.

Integer program (IP): ditto, but with some integer-valued variables

Standard Maximum Problem

Normal form for maximization problems:

Definition (Standard Maximum Problem)

Find values for $x_1, \ldots, x_n$, to maximize

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

subject to the constraints

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &\leq b_1 \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &\leq b_2 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &\leq b_m
\end{aligned}$$

and $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$.

Standard Maximum Problem: Matrix and Vectors

A standard maximum problem is often given by

- an m-vector $\mathbf{b} = \langle b_1, \ldots, b_m \rangle^T$,
- an n-vector $\mathbf{c} = \langle c_1, \ldots, c_n \rangle^T$,
- and an $m \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12}x_2 & \ldots & a_{1n} \\ a_{21} & a_{22}x_2 & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

- Then the problem is to find a vector $\mathbf{x} = \langle x_1, \ldots, x_n \rangle^T$ to maximize $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Standard Minimum Problem

Normal form for minimization problems:

Definition (Standard Minimum Problem)

Find values for $y_1, \ldots, y_m$, to minimize

$$b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

subject to the constraints

$$\begin{aligned} y_1 a_{11} + y_2 a_{21} + \cdots + y_m a_{m1} &\geq c_1 \\ y_1 a_{12} + y_2 a_{22} + \cdots + y_m a_{m2} &\geq c_2 \\ &\vdots \\ y_1 a_{1n} + y_2 a_{2n} + \cdots + y_m a_{mn} &\geq c_n \end{aligned}$$

and $y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0$.

Standard Minimum Problem: Matrix and Vectors

- A standard minimum problem is defined by the same matrix $\mathbf{A}$ and vectors $\mathbf{b}, \mathbf{c}$ as a maximum problem.
- The problem is to find a vector $\mathbf{y} = \langle y_1, \ldots, y_m \rangle^T$ to minimize $\mathbf{y}^T\mathbf{b}$ subject to $\mathbf{y}^T\mathbf{A} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Terminology

- A vector $\mathbf{x}$ for a maximum problem or $\mathbf{y}$ for a minimum problem is feasible if it satisfies the constraints.
- A linear program is feasible if there is such a feasible vector. Otherwise it is infeasible.
- A feasible maximum (resp. minimum) problem is unbounded if the objective function can assume arbitrarily large positive (resp. negative) values at feasible vectors. Otherwise it it bounded.
- The objective value of a bounded feasible maximum (resp. minimum) problem is the maximum (resp. minimum) value of the objective function with a feasible vector.

Standard Problems are a Normal Form

All linear programs can be converted into a standard maximum problem:

- To transform a minimum problem into a maximum problem, multiply the objective function by -1.
- Transform constraints $a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$ to $(-a_{i1})x_1 + \cdots + (-a_{in})x_n \leq -b_i$.
- Solve equality constraints $a_{i1}x_1 + \cdots + a_{in}x_n = b_i$ for some x_j with $a_{ij} \neq 0$ and substitute this solution wherever x_j appears.
- If a variable x can be negative, introduce variables $x' \geq 0$ and $x'' \geq 0$ and replace x everywhere with $x' - x''$.

Solving Linear Programs and Integer Programs

Complexity:

- LP solving is a polynomial-time problem.
- Finding solutions for IPs is NP-complete.

Common idea:

- Approximate IP solution with corresponding LP (LP relaxation).

LP Relaxation

Theorem (LP Relaxation)

The LP relaxation of an integer program is the problem that arises by removing the requirement that variables are integer-valued.

For a maximization (resp. minimization) problem, the objective value of the LP relaxation is an upper (resp. lower) bound on the value of the IP.

Proof idea.

Every feasible vector for the IP is also feasible for the LP. □

Duality

Some LP Theory: Duality

Some LP theory: Every LP has an alternative view (its dual).

- roughly: variables and constraints swap roles
- dual of maximization LP is minimization LP and vice versa
- dual of dual: original LP

Dual Problem

Definition (Dual Problem)

The dual of the standard maximum problem

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}$$

is the minimum problem

$$\text{minimize } \mathbf{y}^T\mathbf{b} \text{ subject to } \mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T \text{ and } \mathbf{y} \geq \mathbf{0}$$

Dual for Diet Problem

Example (Dual of Linear Program for Diet Problem)

maximize $\sum_{j=1}^{m} y_j r_j$ subject to

$$\sum_{j=1}^{m} a_{ij} y_j \leq c_i \qquad \text{for } 1 \leq i \leq n$$

$$y_j \geq 0 \qquad \text{for } 1 \leq j \leq m$$

Duality Theorem

Theorem (Duality Theorem)

If a standard linear program is bounded feasible, then so is its dual, and their objective values are equal.

(Proof omitted.)

The dual provides a different perspective on a problem.

Dual for Diet Problem: Interpretation

Example (Dual of Linear Program for Diet Problem)

maximize $\sum_{j=1}^{m} y_j r_j$ subject to

$$\sum_{j=1}^{m} a_{ij} y_j \leq c_i \qquad \text{for } 1 \leq i \leq n$$

$$y_j \geq 0 \qquad \text{for } 1 \leq j \leq m$$

Find nutrient prices that maximize total worth of daily nutrients.
The value of nutrients in food F_i may not exceed the cost of F_i.

Summary

Summary

- Linear (and integer) programs consist of an objective function that should be maximized or minimized subject to a set of given linear constraints.
- Finding solutions for integer programs is NP-complete.
- LP solving is a polynomial time problem.
- The dual of a maximization LP is a minimization LP and vice versa.
- The dual of a bounded feasible LP has the same objective value.

Further Reading

The slides in this chapter are based on the following excellent tutorial on LP solving:

Thomas S. Ferguson.
Linear Programming – A Concise Introduction.
UCLA, unpublished document available online.