Foundations of Artificial Intelligence

30. Propositional Logic: Reasoning and Resolution

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Propositional Logic: Overview

Chapter overview: propositional logic

- 29. Basics
- 30. Reasoning and Resolution
- 31. DPLL Algorithm
- 32. Local Search and Outlook

Reasoning

Reasoning: Intuition

Reasoning: Intuition

- Generally, formulas only represent an incomplete description of the world.
- In many cases, we want to know if a formula logically follows from (a set of) other formulas.
- What does this mean?

Reasoning: Intuition

- example: $\varphi = (P \lor Q) \land (R \lor \neg P) \land S$
- S holds in every model of φ.
  What about P, Q and R?

$\rightsquigarrow$ consider all models of φ:

- $I_1 = \{P \mapsto \mathbf{F}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{T}\}$
- $I_2 = \{P \mapsto \mathbf{F}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{T}, S \mapsto \mathbf{T}\}$
- $I_3 = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{F}, R \mapsto \mathbf{T}, S \mapsto \mathbf{T}\}$
- $I_4 = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{T}, S \mapsto \mathbf{T}\}$

Observation

- In all models of φ, the formula $Q \lor R$ holds as well.
- We say: "$Q \lor R$ logically follows from φ."

Reasoning: Formally

Definition (logical consequence)

Let Φ be a set of formulas. A formula ψ logically follows from Φ (in symbols: $\Phi \models \psi$) if all models of Φ are also models of ψ.

In other words: for each interpretation I,
if $I \models \varphi$ for all $\varphi \in \Phi$, then also $I \models \psi$.

Question

How can we automatically compute if $\Phi \models \psi$?

- One possibility: Build a truth table. (How?)
- Are there "better" possibilities that (potentially) avoid generating the whole truth table?

Reasoning: Deduction Theorem

Proposition (deduction theorem)

Let Φ be a finite set of formulas and let ψ be a formula. Then

$$\Phi \models \psi \quad \textit{iff} \quad (\bigwedge_{\varphi \in \Phi} \varphi) \to \psi \textit{ is a tautology.}$$

Reasoning: Deduction Theorem

Proposition (deduction theorem)

Let Φ be a finite set of formulas and let ψ be a formula. Then

$$\Phi \models \psi \quad \textit{iff} \quad \Big(\bigwedge_{\varphi \in \Phi} \varphi\Big) \to \psi \textit{ is a tautology.}$$

Proof.

$\Phi \models \psi$
iff for each interpretation I: if $I \models \varphi$ for all $\varphi \in \Phi$, then $I \models \psi$
iff for each interpretation I: if $I \models \bigwedge_{\varphi \in \Phi} \varphi$, then $I \models \psi$
iff for each interpretation I: $I \not\models \bigwedge_{\varphi \in \Phi} \varphi$ or $I \models \psi$
iff for each interpretation I: $I \models (\bigwedge_{\varphi \in \Phi} \varphi) \to \psi$
iff $(\bigwedge_{\varphi \in \Phi} \varphi) \to \psi$ is tautology □

Reasoning

Consequence of Deduction Theorem

Reasoning can be reduced to testing validity.

Algorithm

Question: Does $\Phi \models \psi$ hold?

1. test if $(\bigwedge_{\varphi \in \Phi} \varphi) \to \psi$ is tautology
2. if yes, then $\Phi \models \psi$, otherwise $\Phi \not\models \psi$

In the following: Can we test for validity "efficiently", i.e., without computing the whole truth table?

Resolution

Sets of Clauses

for the rest of this chapter:

- prerequisite: formulas in conjunctive normal form
- clause represented as a set C of literals
- formula represented as a set Δ of clauses

Example

Let $\varphi = (P \vee Q) \wedge \neg P$.

- φ in conjunctive normal form
- φ consists of clauses $(P \vee Q)$ and $\neg P$
- representation of φ as set of sets of literals: $\{\{P, Q\}, \{\neg P\}\}$

Sets of Clauses (Corner Cases)

Distinguish $\Box$ (empty clause) vs. $\emptyset$ (empty set of clauses).

- $\Box$ represents a disjunction over zero literals:

$$\bigvee_{L \in \emptyset} L = \bot$$

- $\Delta_1 = \{\Box\}$ represents a conjunction over one clause:

$$\bigwedge_{\varphi \in \{\bot\}} \varphi = \bot$$

- $\Delta_2 = \emptyset$ represents a conjunction over zero clauses:

$$\bigwedge_{\varphi \in \emptyset} \varphi = \top$$

Resolution: Idea

Observation

- Testing for validity can be reduced to testing unsatisfiability.
- formula φ valid iff $\neg\varphi$ unsatisfiable

Resolution: Idea

- method to test formula φ for unsatisfiability
- idea: derive new formulas from φ that logically follow from φ
- if empty clause $\square$ can be derived $\rightsquigarrow$ φ unsatisfiable

The Resolution Rule

$$\frac{C_1 \cup \{\ell\}, C_2 \cup \{\bar{\ell}\}}{C_1 \cup C_2}$$

- "From $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$, we can conclude $C_1 \cup C_2$."
- $C_1 \cup C_2$ is resolvent of parent clauses $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$.
- The literals ℓ and $\bar{\ell}$ are called resolution literals,
  the corresponding proposition is called resolution variable.
- resolvent follows logically from parent clauses (Why?)

Example

- resolvent of $\{A, B, \neg C\}$ and $\{A, D, C\}$?
- resolvents of $\{\neg A, B, \neg C\}$ and $\{A, D, C\}$?

Resolution: Derivations

Definition (derivation)

Notation: $R(\Delta) = \Delta \cup \{C \mid C$ is resolvent of two clauses in $\Delta\}$

A clause D can be derived from Δ (in symbols $\Delta \vdash D$) if there is a sequence of clauses $C_1, \ldots, C_n = D$ such that for all $i \in \{1, \ldots, n\}$ we have $C_i \in R(\Delta \cup \{C_1, \ldots, C_{i-1}\})$.

Lemma (soundness of resolution)

If $\Delta \vdash D$, then $\Delta \models D$.

Does the converse direction hold as well (completeness)?

Resolution: Completeness?

The converse of the lemma does not hold in general.

example:

- $\{\{A, B\}, \{\neg B, C\}\} \models \{A, B, C\}$, but
- $\{\{A, B\}, \{\neg B, C\}\} \nvdash \{A, B, C\}$

but: converse holds for special case of empty clause $\square$ (no proof)

Theorem (refutation-completeness of resolution)

Δ is unsatisfiable iff $\Delta \vdash \square$

consequences:

- Resolution is a complete proof method for testing unsatisfiability.
- Resolution can be used for general reasoning by reducing to a test for unsatisfiability.

Example

Let $\Phi = \{P \vee Q, \neg P\}$. Does $\Phi \models Q$ hold?

Solution

- test if $((P \vee Q) \wedge \neg P) \rightarrow Q$ is tautology
- equivalently: test if $((P \vee Q) \wedge \neg P) \wedge \neg Q$ is unsatisfiable
- resulting set of clauses: $\Phi' = \{\{P, Q\}, \{\neg P\}, \{\neg Q\}\}$
- resolving $\{P, Q\}$ with $\{\neg P\}$ yields $\{Q\}$
- resolving $\{Q\}$ with $\{\neg Q\}$ yields $\square$
- observation: empty clause can be derived, hence Φ' unsatisfiable
- consequently $\Phi \models Q$

Resolution: Discussion

- Resolution is a complete proof method to test formulas for unsatisfiability.
- In the worst case, resolution proofs can take exponential time.
- In practice, a strategy which determines the next resolution step is needed.
- In the following chapter, we discuss the DPLL algorithm, which is a combination of backtracking and resolution.

Summary

Summary

- Reasoning: the formula ψ follows from the set of formulas Φ if all models of Φ are also models of ψ.
- Reasoning can be reduced to testing validity (with the deduction theorem).
- Testing validity can be reduced to testing unsatisfiability.
- Resolution is a refutation-complete proof method applicable to formulas in conjunctive normal form.

$\rightsquigarrow$ can be used to test if a set of clauses is unsatisfiable