Foundations of Artificial Intelligence

26. Constraint Satisfaction Problems: Path Consistency

Thomas Keller and Florian Pommerening

University of Basel

April 17, 2023

Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems:

- 22.–23. Introduction
- 24.–26. Basic Algorithms
  - 24. Backtracking
  - 25. Arc Consistency
  - 26. Path Consistency
- 27.–28. Problem Structure

Beyond Arc Consistency

Beyond Arc Consistency: Path Consistency

idea of arc consistency:

- For every assignment to a variable u
  there must be a suitable assignment to every other variable v.
- If not: remove values of u for which
  no suitable “partner” assignment to v exists.

$\rightsquigarrow$ tighter unary constraint on u

Beyond Arc Consistency: Path Consistency

idea of arc consistency:

- For every assignment to a variable u there must be a suitable assignment to every other variable v.
- If not: remove values of u for which no suitable “partner” assignment to v exists.

$\rightsquigarrow$ tighter unary constraint on u

This idea can be extended to three variables (path consistency):

- For every joint assignment to variables u, v there must be a suitable assignment to every third variable w.
- If not: remove pairs of values of u and v for which no suitable “partner” assignment to w exists.

$\rightsquigarrow$ tighter binary constraint on u and v

Beyond Arc Consistency: i-Consistency

general concept of i-consistency for $i \geq 2$:

- For every joint assignment to variables $v_1, \dots, v_{i-1}$ there must be a suitable assignment to every i-th variable v_i.
- If not: remove value tuples of $v_1, \dots, v_{i-1}$ for which no suitable "partner" assignment for v_i exists.

$\rightsquigarrow$ tighter $(i-1)$-ary constraint on $v_1, \dots, v_{i-1}$

Beyond Arc Consistency: i-Consistency

general concept of i-consistency for $i \geq 2$:

- For every joint assignment to variables $v_1, \ldots, v_{i-1}$ there must be a suitable assignment to every i-th variable v_i.
- If not: remove value tuples of $v_1, \ldots, v_{i-1}$ for which no suitable "partner" assignment for v_i exists.

$\rightsquigarrow$ tighter $(i-1)$-ary constraint on $v_1, \ldots, v_{i-1}$

- 2-consistency = arc consistency
- 3-consistency = path consistency (*)

  (*) usual definitions differ when ternary constraints are allowed
- i-consistency for $i > 3$
  - rarely used, requires higher-arity constraints
    $\rightsquigarrow$ not considered here

Path Consistency

Path Consistency: Definition

Definition (path consistent)

Let $\mathcal{C} = \langle V, \text{dom}, (R_{uv}) \rangle$ be a constraint network.

(a) Two different variables $u, v \in V$ are path consistent with respect to a third variable $w \in V$ if for all values $d_u \in \text{dom}(u), d_v \in \text{dom}(v)$ with $\langle d_u, d_v \rangle \in R_{uv}$ there is a value $d_w \in \text{dom}(w)$ with $\langle d_u, d_w \rangle \in R_{uw}$ and $\langle d_v, d_w \rangle \in R_{vw}$.

(b) The constraint network $\mathcal{C}$ is path consistent if for any three variables u, v, w, the variables u and v are path consistent with respect to w.

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
$R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$

Are w and y path consistent with respect to z?

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
$R_{yz} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$

$$R_{wy} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$$

Are w and y path consistent with respect to z?

Path Consistency on Running Example

Running Example

$$R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$$
$$R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$$
$$R_{wy} = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle, \langle 3, 4\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle, \langle 4, 4\rangle\}$$

Are w and y path consistent with respect to z? No!

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$
$R_{yz} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$
$R_{wy} = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,4\rangle\}$

Are w and y path consistent with respect to z?

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle\}$
$R_{yz} = \{\langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle, \langle 4, 2 \rangle, \langle 4, 3 \rangle\}$
$R_{wy} = \{\langle 1, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 4 \rangle\}$

Are w and y path consistent with respect to z? Yes!

Path Consistency on Running Example

Running Example

$R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
$R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$
$R_{wy} = \{\langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 4\rangle\}$

Are w and y path consistent with respect to z? Yes!

Path Consistency on Running Example

Running Example

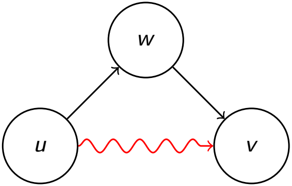
$R_{wz} = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle\}$
$R_{yz} = \{\langle 2, 1\rangle, \langle 3, 1\rangle, \langle 3, 2\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle\}$
$R_{wy} = \{\langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 4\rangle\}$

Are w and y path consistent with respect to z? Yes!

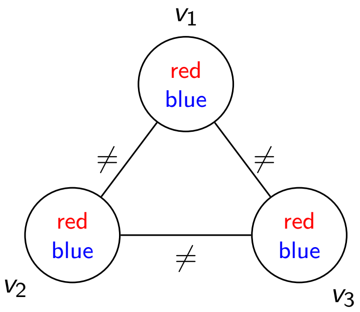
Path Consistency: Remarks

remarks:

- Even if the constraint R_{uv} is trivial, path consistency can infer nontrivial constraints between u and v.
- name "path consistency":
  path $u \to w \to v$ leads to new information on $u \to v$



Path Consistency: Example



arc consistent, but not path consistent

Processing Variable Triples: revise-3

analogous to revise for arc consistency:

function revise-3($\mathcal{C}, u, v, w$):

$\langle V, \text{dom}, (R_{uv})\rangle := \mathcal{C}$

for each $\langle d_u, d_v \rangle \in R_{uv}$:
  if there is no $d_w \in \text{dom}(w)$ with
    $\langle d_u, d_w \rangle \in R_{uw}$ **and** $\langle d_v, d_w \rangle \in R_{vw}$:
      remove $\langle d_u, d_v \rangle$ from R_{uv}

input: constraint network $\mathcal{C}$ and three variables u, v, w of $\mathcal{C}$

effect: u, v path consistent with respect to w.
All violating pairs are removed from R_{uv}.

time complexity: $O(k^3)$ where k is maximal domain size

Enforcing Path Consistency: PC-2

analogous to AC-3 for arc consistency:

function PC-2($\mathcal{C}$):

$\langle V, \mathrm{dom}, (R_{uv})\rangle := \mathcal{C}$

queue $:= \emptyset$
for each set of two variables $\{u, v\}$:
    for each $w \in V \setminus \{u, v\}$:
        insert $\langle u, v, w\rangle$ into *queue*

while *queue* $\neq \emptyset$:
    remove any element $\langle u, v, w\rangle$ from *queue*
    revise-3($\mathcal{C}, u, v, w$)
    if R_{uv} changed in the call to revise-3:
        for each $w' \in V \setminus \{u, v\}$:
            insert $\langle w', u, v\rangle$ into *queue*
            insert $\langle w', v, u\rangle$ into *queue*

PC-2: Discussion

The comments for AC-3 hold analogously.

- PC-2 enforces path consistency
- proof idea: invariant of the **while** loop:
  if $\langle u, v, w \rangle \notin queue$, then u, v path consistent
  with respect to w
- time complexity $O(n^3 k^5)$ for n variables and maximal domain size k (Why?)

Summary

Summary

- generalization of
  arc consistency (considers pairs of variables)
  to path consistency (considers triples of variables)
  and i-consistency (considers i-tuples of variables)
- arc consistency tightens unary constraints
- path consistency tightens binary constraints
- i-consistency tightens $(i-1)$-ary constraints
- higher levels of consistency more powerful
  but more expensive than arc consistency