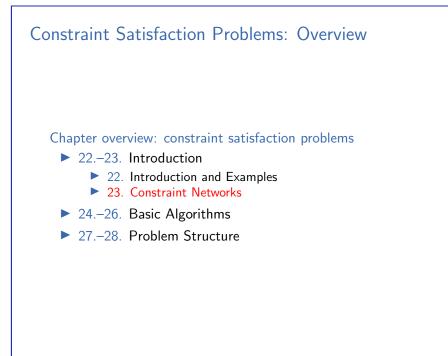
Foundations of Artificial Intelligence 23. Constraint Satisfaction Problems: Constraint Networks

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23. Constraint Satisfaction Problems: Constraint Networks

Constraint Networks

23.1 Constraint Networks

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Constraint Networks: Informally

Constraint Networks: Informal Definition A constraint network is defined by

- a finite set of variables
- ► a finite domain for each variable
- a set of constraints (here: binary relations)

The objective is to find a solution for the constraint network, i.e., an assignment of the variables that complies with all constraints.

Informally, people often just speak of constraint satisfaction problems (CSP) instead of constraint networks.

More formally, a "CSP" is the algorithmic problem of finding a solution for a constraint network.

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Constraint Networks

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Constraint Networks

Variables and Domains

Running Example (informal)

- assign a value from $\{1, 2, 3, 4\}$ to the variables w and y
- and from $\{1, 2, 3\}$ to x and z
- ▶ such that ...

Running Example (formal)

 $\mathcal{C} = \langle V, \mathsf{dom}, (R_{uv}) \rangle$ with

$$\blacktriangleright V = \{w, x, y, z\}$$

• dom(w) = dom(y) = $\{1, 2, 3, 4\}$

•
$$dom(x) = dom(z) = \{1, 2, 3\}$$

▶ ...

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Constraint Networks: Formally

Definition (binary constraint network)

A (binary) constraint network

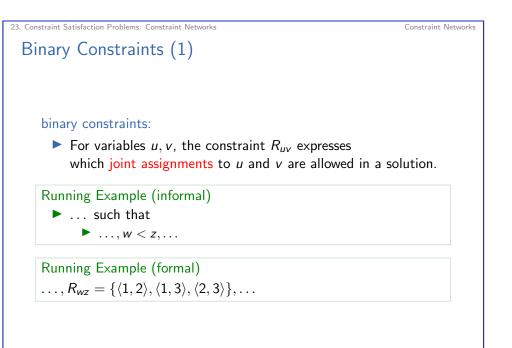
is a 3-tuple $C = \langle V, \text{dom}, (R_{uv}) \rangle$ such that:

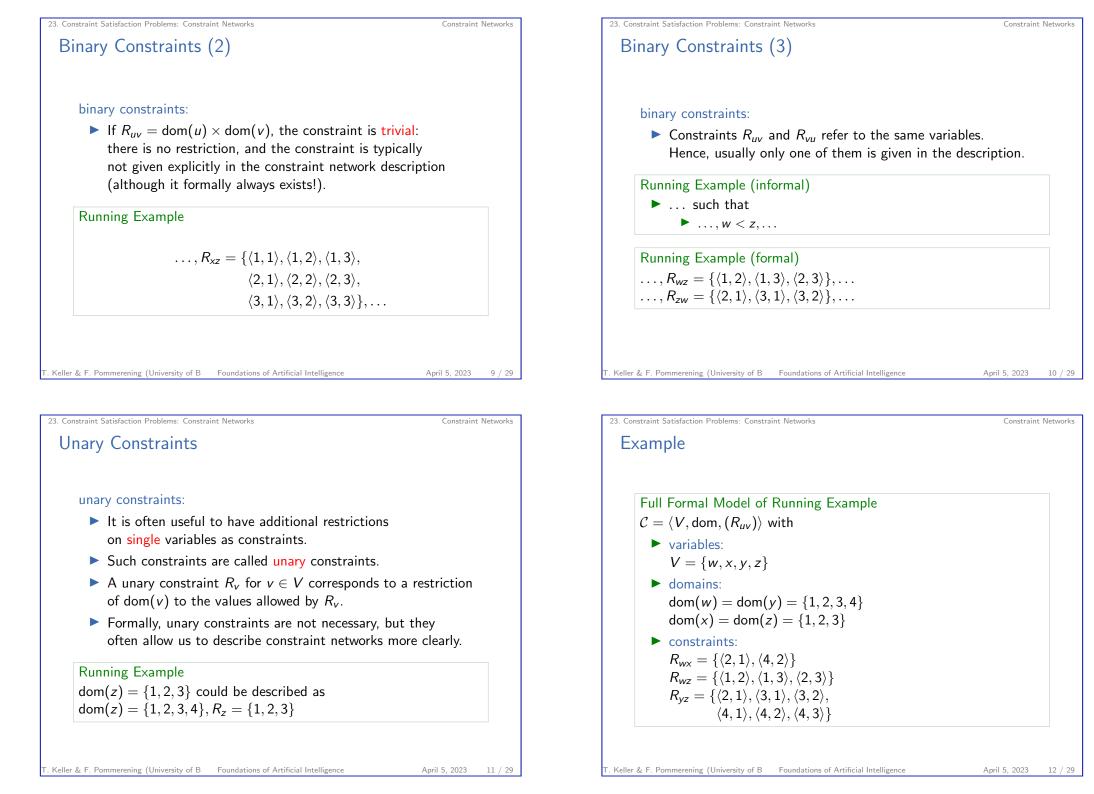
- ► V is a non-empty and finite set of variables,
- dom is a function that assigns a non-empty and finite domain to each variable $v \in V$, and
- ► $(R_{uv})_{u,v \in V, u \neq v}$ is a family of binary relations (constraints) over V where for all $u \neq v$: $R_{uv} \subseteq \text{dom}(u) \times \text{dom}(v)$

possible generalizations:

- ▶ infinite domains (e.g., dom(v) = \mathbb{Z})
- constraints of higher arity (e.g., satisfiability in propositional logic)

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Constraint Networks

Compact Encodings and General Constraint Solvers

Constraint networks allow for compact encodings of large sets of assignments:

- Consider a network with n variables with domains of size k.
- $\rightsquigarrow k^n$ assignments
- For the description as constraint network, at most ⁿ₂, i.e., O(n²) constraints have to be provided. Every constraint in turn consists of at most O(k²) pairs.
- \rightsquigarrow encoding size $O(n^2k^2)$
- We observe: The number of assignments is exponentially larger than the description of the constraint network.
- As a consequence, such descriptions can be used as inputs of general constraint solvers.

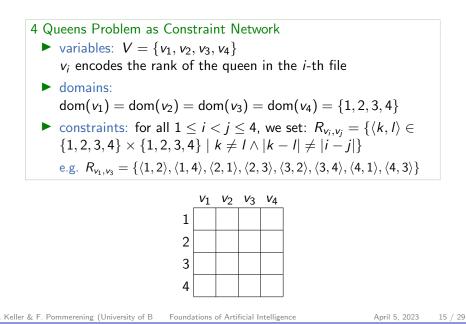
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Example

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23.2 Examples

23. Constraint Satisfaction Problems: Constraint Networks

23. Constraint Satisfaction Problems: Constraint Networks Example: Sudoku Sudoku as Constraint Network • variables: $V = \{v_{ij} \mid 1 \le i, j \le 9\}$; v_{ij} : Value row *i*, column *j* • domains: dom $(v) = \{1, ..., 9\}$ for all $v \in V$ • unary constraints: $R_{v_{ij}} = \{k\}$, if $\langle i, j \rangle$ is a cell with predefined value *k* • binary constraints: for all $v_{ij}, v_{i'j'} \in V$, we set $R_{v_{ij}v_{i'j'}} = \{\langle a, b \rangle \in \{1, ..., 9\} \times \{1, ..., 9\} \mid a \ne b\}$, if i = i' (same row), or j = j' (same column), or $\langle \lceil \frac{i}{3} \rceil, \lceil \frac{j}{3} \rceil \rangle = \langle \lceil \frac{i'}{3} \rceil, \lceil \frac{j'}{3} \rceil \rangle$ (same block)



Example

23.3 Assignments and Consistency

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Assignments and Consistency

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Example

Partial Assignments of Running Example $\alpha_1 = \{ w \mapsto 1, z \mapsto 2 \}$ $\alpha_2 = \{ w \mapsto 3, x \mapsto 1 \}$

Total Assignments of Running Example $\alpha_3 = \{ w \mapsto 1, x \mapsto 1, y \mapsto 2, z \mapsto 2 \}$ $\alpha_4 = \{ w \mapsto 2, x \mapsto 1, y \mapsto 4, z \mapsto 3 \}$

Assignments

Definition (assignment, partial assignment) Let $C = \langle V, \text{dom}, (R_{uv}) \rangle$ be a constraint network. A partial assignment of C (or of V) is a function $\alpha : V' \rightarrow \bigcup_{v \in V} \text{dom}(v)$ with $V' \subseteq V$ and $\alpha(v) \in \text{dom}(v)$ for all $v \in V'$. If V' = V, then α is also called total assignment (or assignment). \rightsquigarrow partial assignments assign values to some or to all variables \rightsquigarrow (total) assignments are defined on all variables

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Consistency

Definition (inconsistent, consistent, violated)

A partial assignment α of a constraint network C is called inconsistent if there are variables u, v such that α is defined for both u and v, and $\langle \alpha(u), \alpha(v) \rangle \notin R_{uv}$.

In this case, we say α violates the constraint R_{uv} .

A partial assignment is called **consistent** if it is not inconsistent.

trivial example: The empty assignment is always consistent.

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Assignments and Consistency

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Assignments and Consistency

Example

Consistent Partial Assignment $\alpha_1 = \{ w \mapsto 1, z \mapsto 2 \}$

Inconsistent Partial Assignment $\alpha_2 = \{ w \mapsto 2, x \mapsto 2 \}$ violates $R_{wx} = \{ \langle 2, 1 \rangle, \langle 4, 2 \rangle \}$

Inconsistent Assignment $\alpha_3 = \{ w \mapsto 2, x \mapsto 1, y \mapsto 2, z \mapsto 2 \}$ violates $R_{wz} = \{ \langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 2, 3 \rangle \}$ and $R_{yz} = \{ \langle 2, 1 \rangle, \langle 3, 1 \rangle, \langle 3, 2 \rangle, \langle 4, 1 \rangle, \langle 4, 2 \rangle, \langle 4, 3 \rangle \}$

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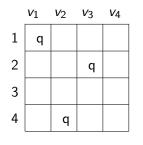
Assignments and Consistency

Consistency vs. Solvability

Note: Consistent partial assignments α cannot necessarily be extended to a solution.

It only means that so far (i.e., on the variables where α is defined) no constraint is violated.

Trivial Example: empty assignment on unsolvable problem Example (4 queens problem): $\alpha = \{v_1 \mapsto 1, v_2 \mapsto 4, v_3 \mapsto 2\}$



Definition (solution, solvable) Let C be a constraint network. A consistent and total assignment of C is called a solution of C. If a solution of C exists, C is called solvable. If no solution exists, C is called inconsistent. Solution of the Running Example $\alpha = \{w \mapsto 2, x \mapsto 1, y \mapsto 4, z \mapsto 3\}$

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Solution

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Assignments and Consistency

Complexity of Constraint Satisfaction Problems

Proposition (CSPs are NP-complete) It is an NP-complete problem to decide whether a given constraint network is solvable.

Proof

Membership in NP:

Guess and check: guess a solution and check it for validity. This can be done in polynomial time in the size of the input.

NP-hardness:

The graph coloring problem is a special case of CSPs and is already known to be NP-complete.

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Tightness of Constraint Networks

Equivalence of Constraint Networks

Definition (tighter, strictly tighter)

Let $C = \langle V, \text{dom}, R_{uv} \rangle$ and $C' = \langle V, \text{dom}', R'_{uv} \rangle$ be constraint networks with equal variable sets V.

C is called tighter than C', in symbols $C \sqsubseteq C'$, if

▶ dom(v) ⊆ dom'(v) for all $v \in V$

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▶ $R_{uv} \subseteq R'_{uv}$ for all $u, v \in V$ (including trivial constraints).

If at least one of these subset equations is strict, then C is called strictly tighter than C', in symbols $C \sqsubset C'$.

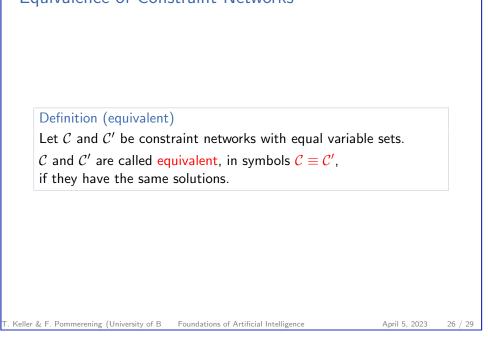
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Outline and Summary

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23.4 Outline and Summary



23. Constraint Satisfaction Problems: Constraint Networks

CSP Algorithms

In the following chapters, we will consider solution algorithms for constraint networks.

basic concepts:

- search: check partial assignments systematically
- backtracking: discard inconsistent partial assignments
- inference: derive equivalent, but tighter constraints to reduce the size of the search space

Outline and Summary

23. Constraint Satisfaction Problems: Constraint Networks

Outline and Summary

Summary

 formal definition of constraint networks: variables, domains, constraints

- compact encodings of exponentially many configurations
- unary and binary constraints
- assignments: partial and total
- consistency of assignments; solutions
- deciding solvability is NP-complete
- tightness of constraints
- equivalence of constraints

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