Foundations of Artificial Intelligence

22. Constraint Satisfaction Problems: Introduction and Examples

Thomas Keller and Florian Pommerening

University of Basel

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Classification

Classification:

Constraint Satisfaction Problems

environment:

- static vs. dynamic
- deterministic vs. non-deterministic vs. stochastic
- fully vs. partially vs. not observable
- discrete vs. continuous
- single-agent vs. multi-agent

problem solving method:

- problem-specific vs. general vs. learning

Special case of a pure search combinatorial optimization problem

Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- 22.–23. Introduction
  - 22. Introduction and Examples
  - 23. Constraint Networks
- 24.–26. Basic Algorithms
- 27.–28. Problem Structure

Introduction

Constraints

What is a Constraint?

a condition that every solution to a problem must satisfy

Examples: Where do constraints occur?

- mathematics: requirements on solutions of optimization problems (e.g., equations, inequalities)
- software testing: specification of invariants to check data consistency (e.g., assertions)
- databases: integrity constraints

Constraint Satisfaction Problems: Informally

Given:

- set of variables with corresponding domains
- set of constraints that the variables must satisfy
  - most commonly binary, i.e., every constraint refers to two variables

Solution:

- assignment to the variables that satisfies all constraints

Examples

Examples

Examples

- 8 queens problem
- Latin squares
- Sudoku
- graph coloring
- satisfiability in propositional logic

more complex examples:

- systems of equations and inequalities
- database queries

Example: 8 Queens Problem (Reminder)

(reminder from previous two chapters)

8 Queens Problem

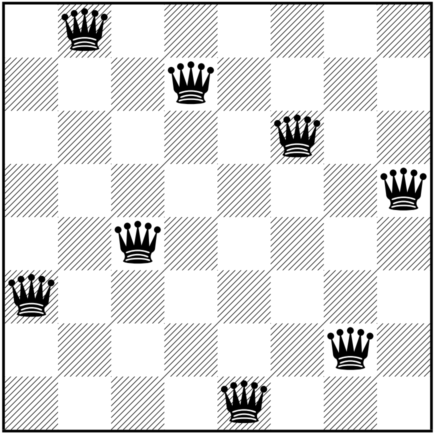
How can we

- place 8 queens on a chess board
- such that no two queens threaten each other?

- originally proposed in 1848
- variants: board size; other pieces; higher dimension

There are 92 solutions, or 12 solutions if we do not count symmetric solutions (under rotation or reflection) as distinct.

8 Queens Problem: Example Solution



example solution for the 8 queens problem

Example: Latin Squares

Latin Squares

How can we

- build an $n \times n$ matrix
  with n symbols
- such that every symbol occurs exactly once
  in every row and every column?

$$\begin{bmatrix}1\end{bmatrix} \quad \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 3 & 1 & 2\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\\ 3 & 4 & 1 & 2\\ 4 & 1 & 2 & 3\end{bmatrix}$$

There exist 12 different Latin squares of size 3,
576 of size 4, 161 280 of size 5, . . . ,
5 524 751 496 156 892 842 531 225 600 of size 9.

Example: Sudoku

Sudoku

How can we

- completely fill an already partially filled 9×9 matrix with numbers between 1–9
- such that each row, each column, and each of the nine 3×3 blocks contains every number exactly once?

2	5			3		9		1
	1				4			
4		7				2		8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9		3				6		4

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6	1	9	8	2	4	3	5	7
4	3	7	9	1	5	2	6	8
3	9	5	2	7	1	4	8	6
7	6	2	4	9	8	1	3	5
8	4	1	6	5	3	7	2	9
1	8	4	3	6	9	5	7	2
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relationship to Latin squares?

Sudoku: Trivia

- well-formed Sudokus have exactly one solution
- to achieve well-formedness, ≥ 17 cells must be filled already (McGuire et al., 2012)
- 6 670 903 752 021 072 936 960 solutions
- only 5 472 730 538 “non-symmetrical” solutions

Example: Graph Coloring

Graph Coloring

How can we

- color the vertices of a given graph using k colors
- such that two neighboring vertices never have the same color?

(The graph and k are problem parameters.)

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Relationship to Sudoku?

Four Color Problem

famous problem in mathematics: Four Color Problem

- Is it always possible to color a planar graph with 4 colors?
- conjectured by Francis Guthrie (1852)
- 1890 first proof that 5 colors suffice
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- several wrong proofs surviving for over 10 years
- solved by Appel and Haken in 1976: 4 colors suffice
- Appel and Haken reduced the problem to 1936 cases, which were then checked by computers
- first famous mathematical problem solved (partially) by computers
  $\rightsquigarrow$ led to controversy: is this a mathematical proof?

Numberphile video:
https://www.youtube.com/watch?v=NgbK43jB4rQ

Satisfiability in Propositional Logic

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How can we

- assign truth values (true/false) to a set of propositional variables
- such that a given set of clauses (formulas of the form $X \vee \neg Y \vee Z$) is satisfied (true)?

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- NP-complete (Cook 1971; Levin 1973)
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- clause length bounded by 3 would not be a restriction

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relationship to previous problems (e.g., Sudoku)?

Practical Applications

- There are thousands of practical applications of constraint satisfaction problems.
- This statement is true already for the satisfiability problem of propositional logic.

some examples:

- verification of hardware and software
- timetabling (e.g., generating time schedules, room assignments for university courses)
- assignment of frequency spectra (e.g., broadcasting, mobile phones)

Running Example

Small Math Puzzle (informal description)

- assign a value from $\{1, 2, 3, 4\}$ to the variables w and y
- and from $\{1, 2, 3\}$ to x and z
- such that
    - $w = 2x$,
    - $w < z$ and
    - $y > z$.

We will keep using this example to explain definitions and algorithms in the next chapters.

Summary

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- constraint satisfaction:
  - find assignment for a set of variables
  - with given variable domains
  - that satisfies a given set of constraints.
- examples:
  - 8 queens problem
  - Latin squares
  - Sudoku
  - graph coloring
  - satisfiability in propositional logic
  - many practical applications