

# Foundations of Artificial Intelligence

## 22. Constraint Satisfaction Problems: Introduction and Examples

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April 5, 2023

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## 22.1 Introduction

## 22.2 Examples

## 22.3 Summary

## Classification

Classification:

Constraint Satisfaction Problems  
environment:

- ▶ **static** vs. dynamic
- ▶ **deterministic** vs. non-deterministic vs. stochastic
- ▶ **fully** vs. partially vs. not **observable**
- ▶ **discrete** vs. continuous
- ▶ **single-agent** vs. multi-agent

problem solving method:

- ▶ problem-specific vs. **general** vs. learning

Special case of a **pure search** combinatorial optimization problem

## Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- ▶ 22.–23. Introduction
  - ▶ 22. **Introduction and Examples**
  - ▶ 23. Constraint Networks
- ▶ 24.–26. Basic Algorithms
- ▶ 27.–28. Problem Structure

## 22.1 Introduction

## Constraints

### What is a Constraint?

a condition that every solution to a problem must satisfy

### Examples: Where do constraints occur?

- ▶ **mathematics**: requirements on solutions of optimization problems (e.g., equations, inequalities)
- ▶ **software testing**: specification of invariants to check data consistency (e.g., assertions)
- ▶ **databases**: integrity constraints

## Constraint Satisfaction Problems: Informally

### Given:

- ▶ set of **variables** with corresponding domains
- ▶ set of **constraints** that the variables must satisfy
  - ▶ most commonly **binary**, i.e., every constraint refers to **two** variables

### Solution:

- ▶ **assignment** to the variables that satisfies all constraints

## 22.2 Examples

## Examples

### Examples

- ▶ 8 queens problem
- ▶ Latin squares
- ▶ Sudoku
- ▶ graph coloring
- ▶ satisfiability in propositional logic

### more complex examples:

- ▶ systems of equations and inequalities
- ▶ database queries

## Example: 8 Queens Problem (Reminder)

(reminder from previous two chapters)

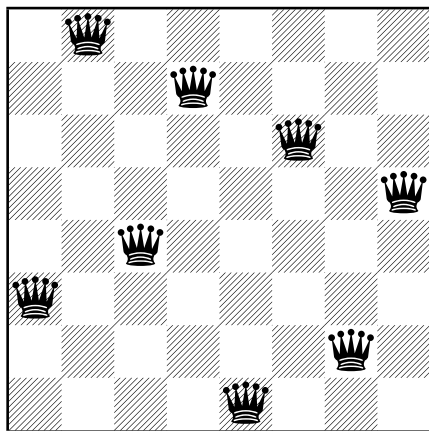
### 8 Queens Problem

How can we

- ▶ place **8 queens** on a chess board
- ▶ such that **no two queens threaten each other?**
- ▶ originally proposed in 1848
- ▶ **variants:** board size; other pieces; higher dimension

There are **92 solutions**, or **12 solutions** if we do not count symmetric solutions (under rotation or reflection) as distinct.

## 8 Queens Problem: Example Solution



example solution for the 8 queens problem

## Example: Latin Squares

### Latin Squares

How can we

- ▶ build an  $n \times n$  **matrix** with  $n$  symbols
- ▶ such that **every symbol occurs exactly once** in every **row** and every **column**?

$$[1] \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

There exist 12 different Latin squares of size 3,  
576 of size 4, 161 280 of size 5, ... ,  
5 524 751 496 156 892 842 531 225 600 of size 9.

## Example: Sudoku

### Sudoku

How can we

- ▶ completely fill an already partially filled  $9 \times 9$  matrix with numbers between 1–9
- ▶ such that each row, each column, and each of the nine  $3 \times 3$  blocks contains every number exactly once?

2	5		3		9	1		
	1			4				
4		7			2			8
		5	2					
				9	8	1		
	4				3			
			3	6			7	2
	7							3
9	3				6			4

relationship to Latin squares?

## Sudoku: Trivia

- ▶ well-formed Sudokus have **exactly one** solution
- ▶ to achieve well-formedness,  $\geq 17$  cells must be filled already (McGuire et al., 2012)
- ▶ 6 670 903 752 021 072 936 960 solutions
- ▶ only 5 472 730 538 “non-symmetrical” solutions

## Example: Graph Coloring

### Graph Coloring

How can we

- ▶ **color the vertices** of a given graph using  $k$  colors
- ▶ such that two neighboring vertices **never have the same** color?

(The graph and  $k$  are problem parameters.)

**NP-complete** problem

- ▶ even for the special case of planar graphs and  $k = 3$
- ▶ easy for  $k = 2$  (also for general graphs)

Relationship to Sudoku?

## Four Color Problem

famous problem in mathematics: **Four Color Problem**

- ▶ Is it always possible to color a **planar** graph with 4 colors?
- ▶ conjectured by Francis Guthrie (1852)
- ▶ 1890 first proof that 5 colors suffice
- ▶ several wrong proofs surviving for over 10 years
- ▶ solved by Appel and Haken in 1976: 4 colors suffice
- ▶ Appel and Haken reduced the problem to 1936 cases, which were then checked by computers
- ▶ first famous mathematical problem solved (partially) by computers
- ~> led to controversy: is this a mathematical proof?

Numberphile video:

<https://www.youtube.com/watch?v=NgBK43jB4rQ>

## Satisfiability in Propositional Logic

### Satisfiability in Propositional Logic

How can we

- ▶ assign **truth values** (true/false) to a set of propositional variables
- ▶ such that a given set of **clauses** (formulas of the form  $X \vee \neg Y \vee Z$ ) is satisfied (true)?

remarks:

- ▶ NP-complete (Cook 1971; Levin 1973)
- ▶ formulas expressed as clauses (instead of arbitrary propositional formulas) is no restriction
- ▶ clause length bounded by 3 would not be a restriction

relationship to previous problems (e.g., Sudoku)?

## Practical Applications

- ▶ There are **thousands** of practical applications of constraint satisfaction problems.
- ▶ This statement is true already for the satisfiability problem of propositional logic.

some examples:

- ▶ verification of hardware and software
- ▶ timetabling (e.g., generating time schedules, room assignments for university courses)
- ▶ assignment of frequency spectra (e.g., broadcasting, mobile phones)

## Running Example

### Small Math Puzzle (informal description)

- ▶ assign a value from  $\{1, 2, 3, 4\}$  to the variables  $w$  and  $y$
- ▶ and from  $\{1, 2, 3\}$  to  $x$  and  $z$
- ▶ such that
  - ▶  $w = 2x$ ,
  - ▶  $w < z$  and
  - ▶  $y > z$ .

We will keep using this example to explain definitions and algorithms in the next chapters.

## 22.3 Summary

## Summary

- ▶ **constraint satisfaction:**
  - ▶ find **assignment** for a set of **variables**
  - ▶ with given **variable domains**
  - ▶ that satisfies a given set of **constraints**.
- ▶ **examples:**
  - ▶ 8 queens problem
  - ▶ Latin squares
  - ▶ Sudoku
  - ▶ graph coloring
  - ▶ satisfiability in propositional logic
  - ▶ many practical applications