Foundations of Artificial Intelligence

20. Combinatorial Optimization: Introduction and Hill-Climbing

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20.1 Combinatorial Optimization

Introduction

previous chapters: classical state-space search

- find action sequence (path) from initial to goal state
- difficulty: large number of states ("state explosion")

next chapters: combinatorial optimization

$\leadsto$ similar scenario, but:

- no actions or transitions
- don't search for path, but for configuration ("state") with low cost/high quality

Combinatorial Optimization: Example

Example: Nurse Scheduling Problem

- find a schedule for a hospital
- satisfy hard constraints
  - labor laws, hospital policies, ...
  - nurses working night shifts should not work early next day
  - have enough nurses with required skills present at all times
- maximize satisfaction of soft constraints
  - individual preferences, reduce overtime, fair distribution, ...

We are interested in a (high-quality) solution, not a path to a goal.

Combinatorial Optimization: Overview

Chapter overview: combinatorial optimization

- 20. Introduction and Hill-Climbing
- 21. Advanced Techniques

Combinatorial Optimization Problems

Definition (combinatorial optimization problem)

A combinatorial optimization problem (COP)
is given by a tuple $\langle C, S, opt, v\rangle$ consisting of:

- a finite set of (solution) candidates C
- a finite set of solutions $S \subseteq C$
- an objective sense $opt \in \{\min, \max\}$
- an objective function $v : S \to \mathbb{R}$

Remarks:

- "problem" here in another sense (= "instance")
  than commonly used in computer science
- practically interesting COPs usually have
  too many candidates to enumerate explicitly

Optimal Solutions

Definition (optimal)

Let $\mathcal{O} = \langle C, S, opt, v\rangle$ be a COP.

The optimal solution quality v^* of $\mathcal{O}$ is defined as

$$v^* = \begin{cases} \min_{c\in S} v(c) & \text{if } opt = \min \\ \max_{c\in S} v(c) & \text{if } opt = \max \end{cases}$$

(v^* is undefined if $S = \emptyset$.)

A solution s of $\mathcal{O}$ is called optimal if $v(s) = v^*$.

Combinatorial Optimization

The basic algorithmic problem we want to solve:

Combinatorial Optimization

Find a solution of good (ideally, optimal) quality
for a combinatorial optimization problem $\mathcal{O}$
or prove that no solution exists.

Good here means close to v^* (the closer, the better).

Relevance and Hardness

- There is a huge number of practically important combinatorial optimization problems.
- Solving these is a central focus of operations research.
- Many important combinatorial optimization problems are NP-complete.
- Most "classical" NP-complete problems can be formulated as combinatorial optimization problems.

$\rightsquigarrow$ Examples: TSP, VertexCover, Clique, BinPacking, Partition

Search vs. Optimization

Combinatorial optimization problems have

- a search aspect (among all candidates C, find a solution from the set S) and
- an optimization aspect (among all solutions in S, find one of high quality).

Pure Search/Optimization Problems

Important special cases arise when one of the two aspects is trivial:

- pure search problems:
  - all solutions are of equal quality
  - difficulty is in finding a solution at all
  - formally: v is a constant function (e.g., constant 0); *opt* can be chosen arbitrarily (does not matter)
- pure optimization problems:
  - all candidates are solutions
  - difficulty is in finding solutions of high quality
  - formally: $S = C$

20.2 Example

Example: 8 Queens Problem

8 Queens Problem

How can we

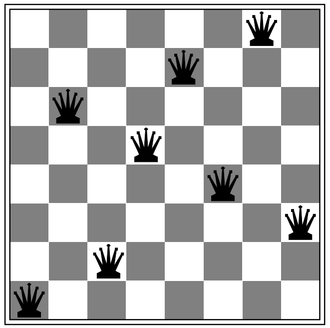
- place 8 queens on a chess board
- such that no two queens threaten each other?

- originally proposed in 1848
- variants: board size; other pieces; higher dimension

There are 92 solutions, or 12 solutions if we do not count symmetric solutions (under rotation or reflection) as distinct.

Example: 8 Queens Problem

Problem: Place 8 queens on a chess board
such that no two queens threaten each other.



Is this candidate a solution?

Formally: 8 Queens Problem

How can we formalize the problem?

idea:

- obviously there must be exactly one queen in each file ("column")
- describe candidates as 8-tuples, where the i-th entry denotes the rank ("row") of the queen in the i-th file

formally: $\mathcal{O} = \langle C, S, opt, v \rangle$ with

- $C = \{1, \dots, 8\}^8$
- $S = \{\langle r_1, \dots, r_8 \rangle \mid \forall 1 \leq i < j \leq 8 : r_i \neq r_j \wedge |r_i - r_j| \neq |i - j|\}$
- v constant, opt irrelevant (pure search problem)

20.3 Local Search: Hill Climbing

Algorithms for Combinatorial Optimization Problems

How can we algorithmically solve COPs?

- formulation as classical state-space search
  ⇝ previous chapters
- formulation as constraint network ⇝ next session
- formulation as logical satisfiability problem ⇝ later
- formulation as mathematical optimization problem (LP/IP)
  ⇝ not in this course
- local search ⇝ this and next chapter

Search Methods for Combinatorial Optimization

- main ideas of heuristic search applicable for COPs
  ⇝ states ≈ candidates
- main difference: no "actions" in problem definition
  - instead, we (as algorithm designers) can choose which candidates to consider neighbors
  - definition of neighborhood critical aspect of designing good algorithms for a given COP
- "path to goal" irrelevant to the user
  - no path costs, parents or generating actions

  ⇝ no search nodes needed

Local Search: Idea

main ideas of local search algorithms for COPs:

- heuristic h estimates quality of candidates
  - for pure optimization: often objective function v itself
  - for pure search: often distance estimate to closest solution (as in state-space search)
- do not remember paths, only candidates
- often only one current candidate ⇝ very memory-efficient (however, not complete or optimal)
- often initialization with random candidate
- iterative improvement by hill climbing

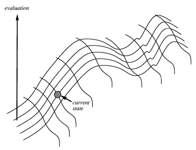
Hill Climbing

Hill Climbing (for Maximization Problems)

current := a random candidate
repeat:
  next := a neighbor of *current* with maximum h value
  if $h(\mathit{next}) \leq h(\mathit{current})$:
    return *current*
  current := *next*

Remarks:

- search as walk "uphill" in a landscape defined by the neighborhood relation
- heuristic values define "height" of terrain
- analogous algorithm for minimization problems also traditionally called "hill climbing" even though the metaphor does not fully fit

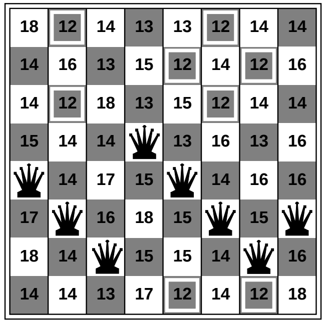


Properties of Hill Climbing

- always terminates (Why?)
- no guarantee that result is a solution
- if result is a solution, it is locally optimal w.r.t. h, but no global quality guarantees

Example: 8 Queens Problem

Problem: Place 8 queens on a chess board such that no two queens threaten each other.
possible heuristic: no. of pairs of queens threatening each other (formalization as minimization problem)
possible neighborhood: move one queen within its file



Performance of Hill Climbing for 8 Queens Problem

- problem has $8^8 \approx 17$ million candidates (reminder: 92 solutions among these)
- after random initialization, hill climbing finds a solution in around 14% of the cases
- only around 3–4 steps on average!

20.4 Summary

Summary

combinatorial optimization problems:

- find solution of good quality (objective value) among many candidates
- special cases:
  - pure search problems
  - pure optimization problems
- differences to state-space search: no actions, paths etc.; only "state" matters

often solved via local search:

- consider one candidate (or a few) at a time; try to improve it iteratively