

Foundations of Artificial Intelligence

18. State-Space Search: Properties of A^* , Part I

Thomas Keller and Florian Pommerening

University of Basel

March 29, 2023

State-Space Search: Overview

Chapter overview: state-space search

- 5.–7. Foundations
- 8.–12. Basic Algorithms
- 13.–19. Heuristic Algorithms
 - 13. Heuristics
 - 14. Analysis of Heuristics
 - 15. Best-first Graph Search
 - 16. Greedy Best-first Search, A^* , Weighted A^*
 - 17. IDA *
 - 18. Properties of A^* , Part I
 - 19. Properties of A^* , Part II

Introduction

Optimality of A^*

- advantage of A^* over greedy search:
 optimal for heuristics with suitable properties
- very important result!

↪ next chapters: a closer look at A^*

- A^* with reopening ↪ this chapter
- A^* without reopening ↪ next chapter

Optimality of A^* with Reopening

In this chapter, we prove that A^* with reopening is optimal when using **admissible** heuristics.

For this purpose, we

- give some basic definitions
- prove two lemmas regarding the behaviour of A^*
- use these to prove the main result

Reminder: A^* with Reopening

reminder: A^* with reopening

A^* with Reopening

```
open := new MinHeap ordered by  $\langle f, h \rangle$ 
if  $h(\text{init}()) < \infty$ :
    open.insert(make_root_node())
distances := new HashMap
while not open.is_empty():
     $n := \text{open.pop\_min}()$ 
    if distances.lookup( $n.\text{state}$ ) = none or  $g(n) < \text{distances}[n.\text{state}]$ :
        distances[ $n.\text{state}$ ] :=  $g(n)$ 
        if is_goal( $n.\text{state}$ ):
            return extract_path( $n$ )
        for each  $\langle a, s' \rangle \in \text{succ}(n.\text{state})$ :
            if  $h(s') < \infty$ :
                 $n' := \text{make\_node}(n, a, s')$ 
                open.insert( $n'$ )
return unsolvable
```

Solvable States

Definition (solvable)

A state s of a state space is called **solvable** if $h^*(s) < \infty$.

Optimal Paths to States

Definition (g^*)

Let s be a state of a state space with initial state s_I .

We write $g^*(s)$ for the cost of the optimal (cheapest) path from s_I to s (∞ if s is unreachable).

Remarks:

- g is defined for nodes, g^* for states (Why?)
- $g^*(n.state) \leq g(n)$ for all nodes n generated by a search algorithm (Why?)

Settled States in A^*

Definition (settled)

A state s is called **settled** at a given point during the execution of A^* (with or without reopening) if s is included in *distances* and $distances[s] = g^*(s)$.

Optimal Continuation Lemma

Optimal Continuation Lemma

We now show the first important result for A^* with reopening:

Lemma (optimal continuation lemma)

Consider A^* *with reopening* using a *safe* heuristic at the beginning of any iteration of the **while** loop.

If

- state s is settled,
- state s' is a solvable successor of s , and
- an optimal path from s_I to s' of the form $\langle s_I, \dots, s, s' \rangle$ exists,

then

- s' is settled or
- open contains a node n' with $n'.\text{state} = s'$ and $g(n') = g^*(s')$.

Optimal Continuation Lemma: Intuition

(Proof follows on the next slides.)

Intuitively, the lemma states:

*If no optimal path to a given state has been found yet,
open must contain a “good” node that contributes
to finding an optimal path to that state.*

(This potentially requires multiple applications of the lemma
along an optimal path to the state.)

Optimal Continuation Lemma: Proof (1)

Proof.

Consider states s and s' with the given properties at the start of some iteration (“iteration A”) of A^* .

Optimal Continuation Lemma: Proof (1)

Proof.

Consider states s and s' with the given properties at the start of some iteration (“iteration A”) of A^* .

Because s is settled, an earlier iteration (“iteration B”) set $distances[s] := g^*(s)$.

Optimal Continuation Lemma: Proof (1)

Proof.

Consider states s and s' with the given properties at the start of some iteration (“iteration A”) of A^* .

Because s is settled, an earlier iteration (“iteration B”) set $distances[s] := g^*(s)$.

Thus iteration B removed a node n with $n.state = s$ and $g(n) = g^*(s)$ from *open*.

Optimal Continuation Lemma: Proof (1)

Proof.

Consider states s and s' with the given properties at the start of some iteration (“iteration A”) of A^* .

Because s is settled, an earlier iteration (“iteration B”) set $distances[s] := g^*(s)$.

Thus iteration B removed a node n with $n.state = s$ and $g(n) = g^*(s)$ from *open*.

A^* did not terminate in iteration B.
(Otherwise iteration A would not exist.)

Hence n was expanded in iteration B.

...

Optimal Continuation Lemma: Proof (2)

Proof (continued).

This expansion considered the successor s' of s .

Because s' is solvable, we have $h^*(s') < \infty$.

Because h is safe, this implies $h(s') < \infty$.

Hence a successor node n' was generated for s' .

Optimal Continuation Lemma: Proof (2)

Proof (continued).

This expansion considered the successor s' of s .

Because s' is solvable, we have $h^*(s') < \infty$.

Because h is safe, this implies $h(s') < \infty$.

Hence a successor node n' was generated for s' .

This node n' satisfies the consequence of the lemma.

Hence the criteria of the lemma were satisfied for s and s' after iteration B.

Optimal Continuation Lemma: Proof (2)

Proof (continued).

This expansion considered the successor s' of s .

Because s' is solvable, we have $h^*(s') < \infty$.

Because h is safe, this implies $h(s') < \infty$.

Hence a successor node n' was generated for s' .

This node n' satisfies the consequence of the lemma.

Hence the criteria of the lemma were satisfied for s and s' after iteration B.

To complete the proof, we show: if the consequence of the lemma is satisfied at the beginning of an iteration, it is also satisfied at the beginning of the next iteration. ...

Optimal Continuation Lemma: Proof (3)

Proof (continued).

- If s' is settled at the beginning of an iteration, it remains settled until termination.

Optimal Continuation Lemma: Proof (3)

Proof (continued).

- If s' is settled at the beginning of an iteration, it remains settled until termination.
- If s' is not yet settled and *open* contains a node n' with $n'.state = s'$ and $g(n') = g^*(s')$ at the beginning of an iteration, then either the node remains in *open* during the iteration, or n' is removed during the iteration and s' becomes settled.



f -Bound Lemma

f-Bound Lemma

We need a second lemma:

Lemma (*f*-bound lemma)

Consider A^* *with reopening* and an *admissible* heuristic applied to a *solvable* state space with optimal solution cost c^* .

Then open contains a node n with $f(n) \leq c^*$ at the beginning of each iteration of the **while** loop.

f-Bound Lemma: Proof (1)

Proof.

Consider the situation at the beginning of any iteration of the **while** loop.

Let $\langle s_0, \dots, s_n \rangle$ with $s_0 := s_I$ be an optimal solution.
(Here we use that the state space is solvable.)

f-Bound Lemma: Proof (1)

Proof.

Consider the situation at the beginning of any iteration of the **while** loop.

Let $\langle s_0, \dots, s_n \rangle$ with $s_0 := s_I$ be an optimal solution.
(Here we use that the state space is solvable.)

Let s_i be the first state in the sequence that is not settled.

(Not all states in the sequence can be settled:
 s_n is a goal state, and when a goal state is inserted into *distances*, A^* terminates.)

...

f-Bound Lemma: Proof (2)

Proof (continued).

Case 1: $i = 0$

Because s_0 is not settled yet, we are at the first iteration of the **while** loop.

f-Bound Lemma: Proof (2)

Proof (continued).

Case 1: $i = 0$

Because s_0 is not settled yet, we are at the first iteration of the **while** loop.

Because the state space is solvable and h is admissible, we have $h(s_0) < \infty$.

f-Bound Lemma: Proof (2)

Proof (continued).

Case 1: $i = 0$

Because s_0 is not settled yet, we are at the first iteration of the **while** loop.

Because the state space is solvable and h is admissible, we have $h(s_0) < \infty$.

Hence *open* contains the root n_0 .

f-Bound Lemma: Proof (2)

Proof (continued).

Case 1: $i = 0$

Because s_0 is not settled yet, we are at the first iteration of the **while** loop.

Because the state space is solvable and h is admissible, we have $h(s_0) < \infty$.

Hence *open* contains the root n_0 .

We obtain: $f(n_0) = g(n_0) + h(s_0) = 0 + h(s_0) \leq h^*(s_0) = c^*$, where “ \leq ” uses the admissibility of h .

This concludes the proof for this case.

...

f-Bound Lemma: Proof (3)

Proof (continued).

Case 2: $i > 0$

Then s_{i-1} is settled and s_i is not settled.

Moreover, s_i is a solvable successor of s_{i-1} and $\langle s_0, \dots, s_{i-1}, s_i \rangle$ is an optimal path from s_0 to s_i .

f-Bound Lemma: Proof (3)

Proof (continued).

Case 2: $i > 0$

Then s_{i-1} is settled and s_i is not settled.

Moreover, s_i is a solvable successor of s_{i-1} and $\langle s_0, \dots, s_{i-1}, s_i \rangle$ is an optimal path from s_0 to s_i .

We can hence apply the optimal continuation lemma (with $s = s_{i-1}$ and $s' = s_i$) and obtain:

- (A) s_i is settled, or
- (B) *open* contains n' with $n'.\text{state} = s_i$ and $g(n') = g^*(s_i)$.

f-Bound Lemma: Proof (3)

Proof (continued).

Case 2: $i > 0$

Then s_{i-1} is settled and s_i is not settled.

Moreover, s_i is a solvable successor of s_{i-1} and $\langle s_0, \dots, s_{i-1}, s_i \rangle$ is an optimal path from s_0 to s_i .

We can hence apply the optimal continuation lemma (with $s = s_{i-1}$ and $s' = s_i$) and obtain:

- (A) s_i is settled, or
- (B) *open* contains n' with $n'.\text{state} = s_i$ and $g(n') = g^*(s_i)$.

Because (A) is false, (B) must be true.

f-Bound Lemma: Proof (3)

Proof (continued).

Case 2: $i > 0$

Then s_{i-1} is settled and s_i is not settled.

Moreover, s_i is a solvable successor of s_{i-1} and $\langle s_0, \dots, s_{i-1}, s_i \rangle$ is an optimal path from s_0 to s_i .

We can hence apply the optimal continuation lemma (with $s = s_{i-1}$ and $s' = s_i$) and obtain:

(A) s_i is settled, or

(B) *open* contains n' with $n'.\text{state} = s_i$ and $g(n') = g^*(s_i)$.

Because (A) is false, (B) must be true.

We conclude: *open* contains n' with

$f(n') = g(n') + h(s_i) = g^*(s_i) + h(s_i) \leq g^*(s_i) + h^*(s_i) = c^*$,
where " \leq " uses the admissibility of h .



Optimality of A^* with Reopening

Optimality of A^* with Reopening

We can now show the main result of this chapter:

Theorem (optimality of A^* with reopening)

A^ with reopening is optimal when using an admissible heuristic.*

Optimality of A^* with Reopening: Proof

Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost c^* where A^* with reopening and an admissible heuristic returns a solution with cost $c > c^*$.

Optimality of A^* with Reopening: Proof

Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost c^* where A^* with reopening and an admissible heuristic returns a solution with cost $c > c^*$.

This means that in the last iteration, the algorithm removes a node n with $g(n) = c > c^*$ from *open*.

Optimality of A^* with Reopening: Proof

Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost c^* where A^* with reopening and an admissible heuristic returns a solution with cost $c > c^*$.

This means that in the last iteration, the algorithm removes a node n with $g(n) = c > c^*$ from *open*.

With $h(n.state) = 0$ (because h is admissible and hence goal-aware), this implies:

Optimality of A^* with Reopening: Proof

Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost c^* where A^* with reopening and an admissible heuristic returns a solution with cost $c > c^*$.

This means that in the last iteration, the algorithm removes a node n with $g(n) = c > c^*$ from *open*.

With $h(n.state) = 0$ (because h is admissible and hence goal-aware), this implies:

$$f(n) = g(n) + h(n.state) = g(n) + 0 = g(n) = c > c^*.$$

Optimality of A^* with Reopening: Proof

Proof.

By contradiction: assume that the theorem is wrong.

Hence there is a state space with optimal solution cost c^* where A^* with reopening and an admissible heuristic returns a solution with cost $c > c^*$.

This means that in the last iteration, the algorithm removes a node n with $g(n) = c > c^*$ from *open*.

With $h(n.\text{state}) = 0$ (because h is admissible and hence goal-aware), this implies:

$$f(n) = g(n) + h(n.\text{state}) = g(n) + 0 = g(n) = c > c^*.$$

A^* always removes a node n with minimal f value from *open*.

With $f(n) > c^*$, we get a contradiction to the f -bound lemma, which completes the proof. □

Summary

Summary

- A^* with reopening using an admissible heuristic is optimal.
- The proof is based on the following lemmas that hold for solvable state spaces and admissible heuristics:
 - **optimal continuation lemma**: The open list always contains nodes that make progress towards an optimal solution.
 - **f -bound lemma**: The minimum f value in the open list at the beginning of each A^* iteration is a lower bound on the optimal solution cost.