Foundations of Artificial Intelligence

10. State-Space Search: Breadth-first Search

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State-Space Search: Overview

Chapter overview: state-space search

- 5.–7. Foundations
- 8.–12. Basic Algorithms
  - 8. Data Structures for Search Algorithms
  - 9. Tree Search and Graph Search
  - 10. Breadth-first Search
  - 11. Uniform Cost Search
  - 12. Depth-first Search and Iterative Deepening
- 13.–19. Heuristic Algorithms

Blind Search

Blind Search

In Chapters 10–12 we consider blind search algorithms:

Blind Search Algorithms

Blind search algorithms use no information
about state spaces apart from the black box interface.

They are also called uninformed search algorithms.

contrast: heuristic search algorithms (Chapters 13–19)

Blind Search Algorithms: Examples

examples of blind search algorithms:

- breadth-first search
- uniform cost search
- depth-first search
- depth-limited search
- iterative deepening search

Blind Search Algorithms: Examples

examples of blind search algorithms:

- breadth-first search ($\rightsquigarrow$ this chapter)
- uniform cost search
- depth-first search
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Blind Search Algorithms: Examples

examples of blind search algorithms:

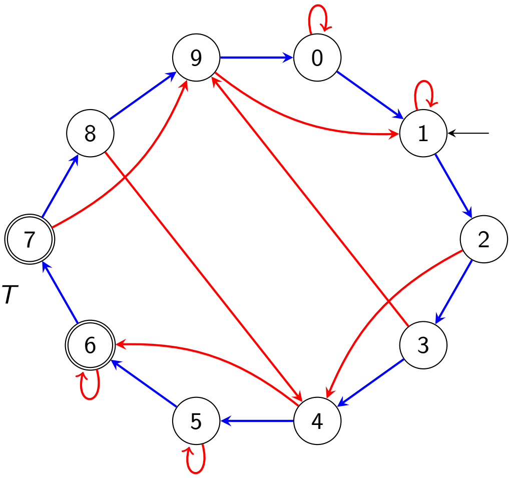
- breadth-first search ($\rightsquigarrow$ this chapter)
- uniform cost search ($\rightsquigarrow$ Chapter 11)
- depth-first search ($\rightsquigarrow$ Chapter 12)
- depth-limited search ($\rightsquigarrow$ Chapter 12)
- iterative deepening search ($\rightsquigarrow$ Chapter 12)

Breadth-first Search: Introduction

Running Example: Reminder

bounded inc-and-square:

- $S = \{0, 1, \ldots, 9\}$
- $A = \{inc, sqr\}$
- $cost(inc) = cost(sqr) = 1$
- T s.t. for $i = 0, \ldots, 9$:
  - $\langle i, inc, (i + 1) \mod 10 \rangle \in T$
  - $\langle i, sqr, i^2 \mod 10 \rangle \in T$
- $s_I = 1$
- $S_\star = \{6, 7\}$



Idea and Example

breadth-first search:

- expands nodes in order of generation (FIFO)
  $\rightsquigarrow$ e.g., open list as linked list or deque
- different variants and optimizations ($\rightsquigarrow$ later)
  - Use a closed list?
  - When to update closed list?
  - When to perform duplicate check?
  - When to perform goal test?

next

open: [1]

closed: { }

Idea and Example

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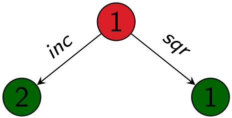

next

open: [2 1]

closed: $\{1\}$

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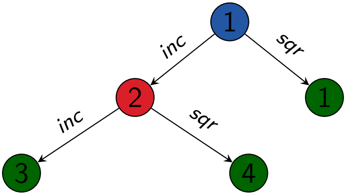

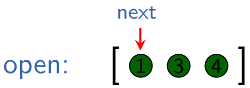

closed: $\{1, 2\}$

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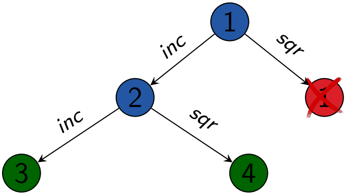

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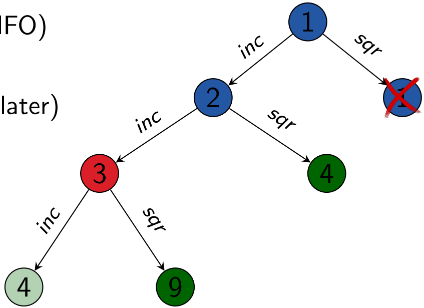

next

open: [4 4 9]

closed: $\{1, 2, 3\}$

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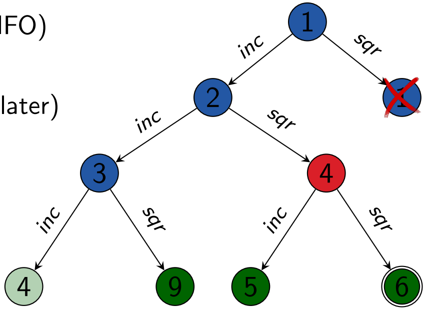

open: [4 9 5 6]

next

closed: $\{1, 2, 3, 4\}$

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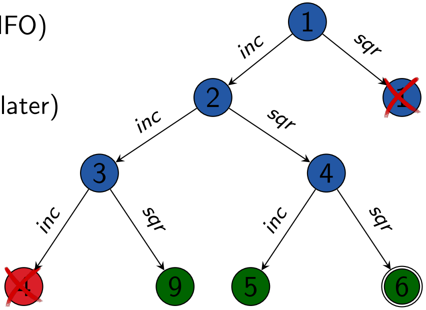

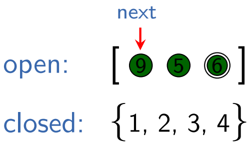

Idea and Example

breadth-first search:

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  ⇝ e.g., open list as linked list or deque
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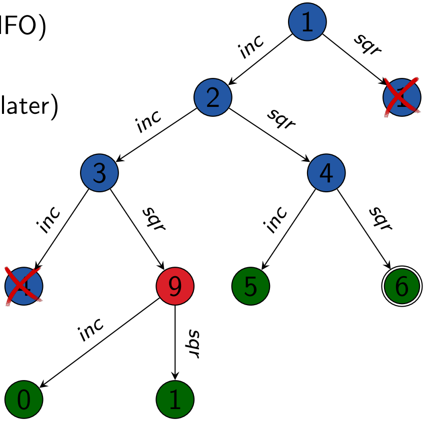

next

open: [5 6 0 1]

closed: $\{1, 2, 3, 4, 9\}$

Idea and Example

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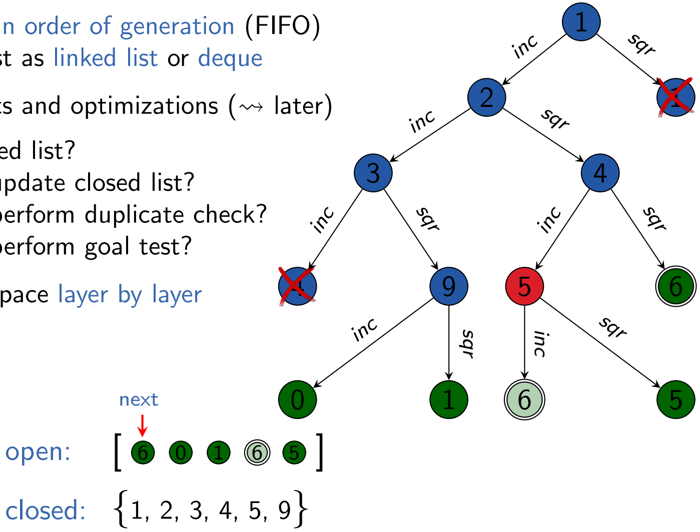

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  - Use a closed list?
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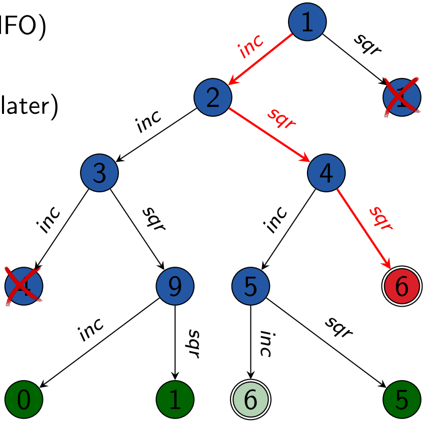
  - When to perform duplicate check?
  - When to perform goal test?
- searches state space layer by layer
- always finds shallowest goal state first



next

open: [0 1 6 5]

closed: $\{1, 2, 3, 4, 5, 6, 9\}$

Breadth-first Search: Tree Search or Graph Search?

Breadth-first search can be performed

- without duplicate elimination (as a tree search)
  $\rightsquigarrow$ BFS-Tree
- or with duplicate elimination (as a graph search)
  $\rightsquigarrow$ BFS-Graph

(BFS = breadth-first search).

$\rightsquigarrow$ We consider both variants.

BFS-Tree

Reminder: Generic Tree Search Algorithm

reminder from Chapter 9:

Generic Tree Search

$open$:= **new** OpenList
$open$.insert(make_root_node())
while not $open$.is_empty():
    n := $open$.pop()
    if is_goal(n.state):
        return extract_path(n)
    for each $\langle a, s' \rangle \in$ succ(n.state):
        n' := make_node(n, a, s')
        $open$.insert(n')
return unsolvable

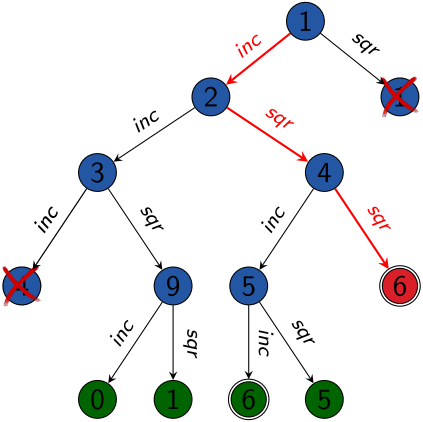
BFS-Tree (1st Attempt)

breadth-first search without duplicate elimination (1st attempt):

BFS-Tree (1st Attempt)

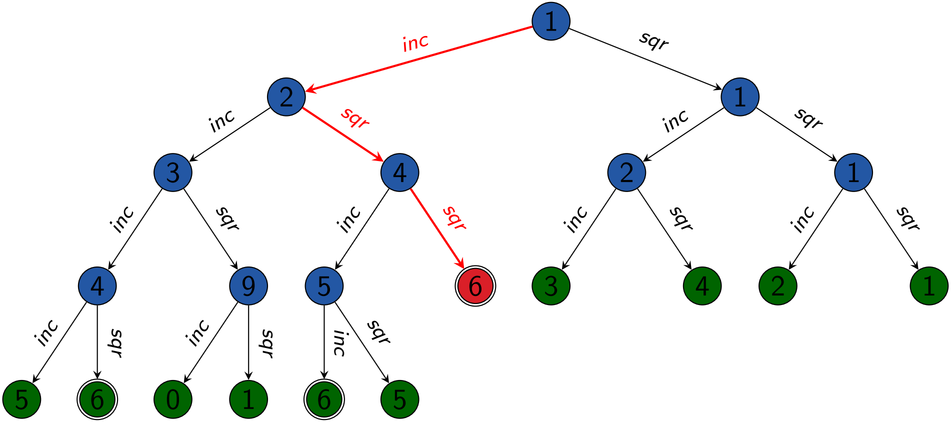
open := **new** Deque
open.push_back(make_root_node())
while not *open*.is_empty():
    n := *open*.pop_front()
    if is_goal(n.state):
        return extract_path(n)
    for each $\langle a, s' \rangle \in$ succ(n.state):
        n' := make_node(n, a, s')
        open.push_back(n')
return unsolvable

Example and Discussion



How does the search tree of the initial example differ if we run BFS-Tree (1st Attempt) instead?

Example and Discussion



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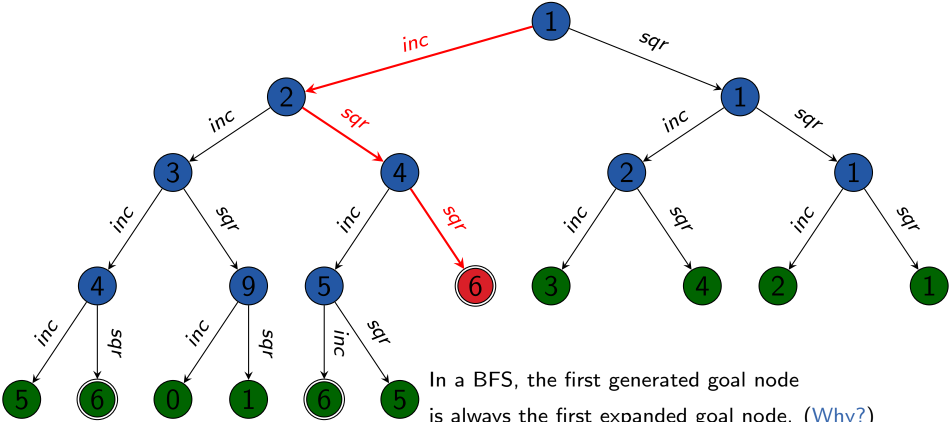
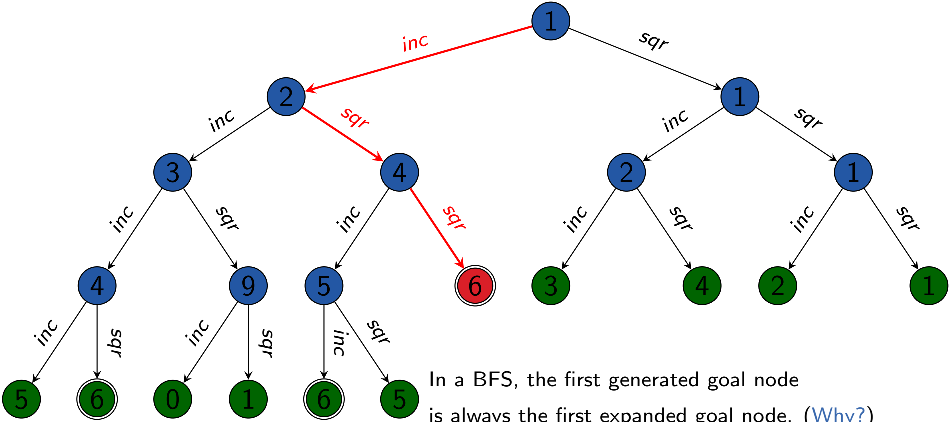
Example and Discussion



In a BFS, the first generated goal node is always the first expanded goal node. (Why?)

Example and Discussion

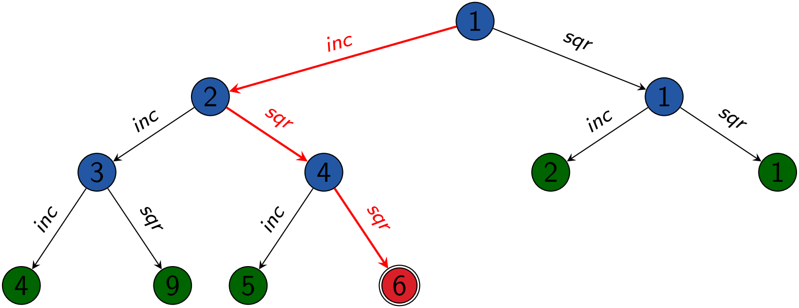


In a BFS, the first generated goal node is always the first expanded goal node. (Why?)

⇝ It is more efficient to perform the goal test upon generating a node (rather than upon expanding it).

⇝ How much effort does this save?

Example and Discussion



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is always the first expanded goal node. (Why?)

$\rightsquigarrow$ It is more efficient to perform the goal test upon generating a node (rather than upon expanding it).

$\rightsquigarrow$ How much effort does this save?

BFS-Tree (2nd Attempt)

breadth-first search without duplicate elimination (2nd attempt):

BFS-Tree (2nd Attempt)

open := **new** Deque
open.push_back(make_root_node())
while not *open*.is_empty():
    n := *open*.pop_front()
    ~~**if** is_goal(n.state):~~
        ~~**return** extract_path(n)~~
    for each $\langle a, s' \rangle \in$ succ(n.state):
        n' := make_node(n, a, s')
        if is_goal(s'):
            return extract_path(n')
        open.push_back(n')
return unsolvable

BFS-Tree (2nd Attempt)

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        n' := make_node(n, a, s')
        if is_goal(s'):
            return extract_path(n')
        open.push_back(n')
return unsolvable

BFS-Tree (2nd Attempt): Discussion

Where is the bug?

BFS-Tree (Final Version)

breadth-first search without duplicate elimination (final version):

BFS-Tree

```
if is_goal(init()):
    return ⟨⟩
open := new Deque
open.push_back(make_root_node())
while not open.is_empty():
    n := open.pop_front()
    for each ⟨a, s′⟩ ∈ succ(n.state):
        n′ := make_node(n, a, s′)
        if is_goal(s′):
            return extract_path(n′)
        open.push_back(n′)
return unsolvable
```

BFS-Graph

Reminder: Generic Graph Search Algorithm

reminder from Chapter 9:

Generic Graph Search

```
open := new OpenList
open.insert(make_root_node())
closed := new ClosedList
while not open.is_empty():
    n := open.pop()
    if closed.lookup(n.state) = none:
        closed.insert(n)
        if is_goal(n.state):
            return extract_path(n)
        for each ⟨a, s'⟩ ∈ succ(n.state):
            n' := make_node(n, a, s')
            open.insert(n')
return unsolvable
```

Adapting Generic Graph Search to Breadth-First Search

adapting the generic algorithm to breadth-first search:

- similar adaptations to BFS-Tree (deque as open list, early goal test)
- as closed list does not need to manage node information, a set data structure suffices
- for the same reasons why early goal tests are a good idea, we should perform duplicate tests against the closed list and updates of the closed lists as early as possible

BFS-Graph (Breadth-First Search with Duplicate Elim.)

BFS-Graph

```
if is_goal(init()):
    return ⟨⟩
open := new Deque
open.push_back(make_root_node())
closed := new HashSet
closed.insert(init())
while not open.is_empty():
    n := open.pop_front()
    for each ⟨a, s'⟩ ∈ succ(n.state):
        n' := make_node(n, a, s')
        if is_goal(s'):
            return extract_path(n')
        if s' ∉ closed:
            closed.insert(s')
            open.push_back(n')
return unsolvable
```

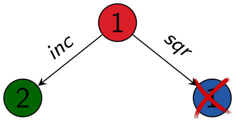
Example

next

open: $[\ 1\ ]$

closed: $\{1\}$

Example

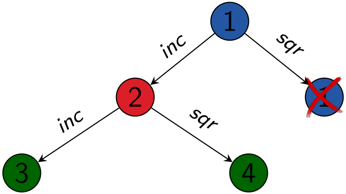


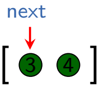
next

open: [2]

closed: $\{1,2\}$

Example

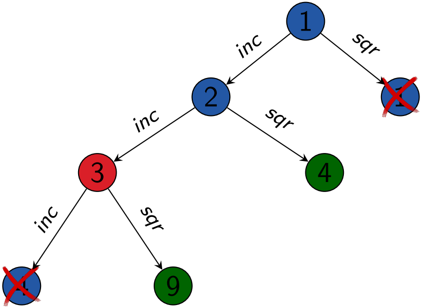




open: [3 4]

closed: $\{1, 2, 3, 4\}$

Example

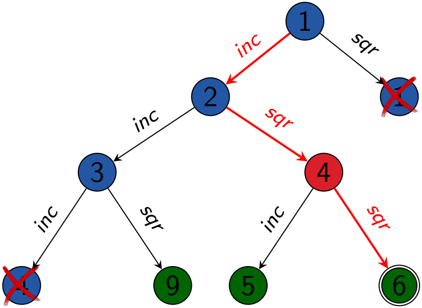


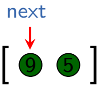
next

open: [4 9]

closed: $\{1, 2, 3, 4, 9\}$

Example





open: [9 5]

closed: $\{1, 2, 3, 4, 5, 6, 9\}$

Properties of Breadth-first Search

Properties of Breadth-first Search

Properties of Breadth-first Search:

- BFS-Tree is semi-complete, but not complete. (Why?)
- BFS-Graph is complete. (Why?)
- BFS (both variants) is optimal
  if all actions have the same cost (Why?),
  but not in general (Why not?).
- complexity: next slides

Breadth-first Search: Complexity

The following result applies to both BFS variants:

Theorem (time complexity of breadth-first search)

Let b be the branching factor and d be the minimal solution length of the given state space. Let $b \geq 2$.

Then the time complexity of breadth-first search is

$$1 + b + b^2 + b^3 + \cdots + b^d = O(b^d)$$

Reminder: we measure time complexity in generated nodes.

It follows that the space complexity of both BFS variants also is $O(b^d)$ (if $b \geq 2$). (Why?)

Breadth-first Search: Example of Complexity

example: $b = 13$; 100 000 nodes/second; 32 bytes/node

d	nodes	time	memory
4	30 940	0.3 s	966 KiB
6	$5.2 \cdot 10^6$	52 s	159 MiB
8	$8.8 \cdot 10^8$	147 min	26 GiB
10	10^{11}	17 days	4.3 TiB
12	10^{13}	8 years	734 TiB
14	10^{15}	1 352 years	121 PiB
16	10^{17}	$2.2 \cdot 10^5$ years	20 EiB
18	10^{20}	$38 \cdot 10^6$ years	3.3 ZiB

Breadth-first Search: Example of Complexity

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Realistic numbers?

d	nodes	time	memory
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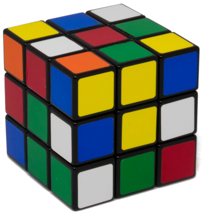

Rubik's cube:

- branching factor: ≈ 13
- typical solution length: 18

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BFS-Tree or BFS-Graph?

What is better, BFS-Tree or BFS-Graph?

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advantages of BFS-Graph:

- complete
- much (!) more efficient if there are many duplicates

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advantages of BFS-Tree:

- simpler
- less overhead (time/space) if there are few duplicates

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- simpler
- less overhead (time/space) if there are few duplicates

Conclusion

BFS-Graph is usually preferable, unless we know that there is a negligible number of duplicates in the given state space.

Summary

Summary

- blind search algorithm: use no information except black box interface of state space
- breadth-first search: expand nodes in order of generation
  - search state space layer by layer
  - can be tree search or graph search
  - complexity $O(b^d)$ with branching factor b, minimal solution length d (if $b \geq 2$)
  - complete as a graph search; semi-complete as a tree search
  - optimal with uniform action costs