

Foundations of Artificial Intelligence

30. Propositional Logic: Reasoning and Resolution

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Propositional Logic: Overview

Chapter overview: propositional logic

- 29. Basics
- 30. Reasoning and Resolution
- 31. DPLL Algorithm
- 32. Local Search and Outlook

Reasoning

Reasoning: Intuition

Reasoning: Intuition

- Generally, formulas only represent an incomplete description of the world.
- In many cases, we want to know if a formula **logically follows** from (a set of) other formulas.
- What does this mean?

Reasoning: Intuition

- **example:** $\varphi = (P \vee Q) \wedge (R \vee \neg P) \wedge S$
- S holds in every model of φ .
What about P , Q and R ?

↪ consider all models of φ :

P	Q	R	S
F	T	F	T
F	T	T	T
T	F	T	T
T	T	T	T

Observation

- In all models of φ , the formula $Q \vee R$ holds as well.
- We say: “ $Q \vee R$ **logically follows** from φ .”

Reasoning: Formally

Definition (logical consequence)

Let Φ be a set of formulas. A formula ψ **logically follows** from Φ (in symbols: $\Phi \models \psi$) if all models of Φ are also models of ψ .

German: logische Konsequenz, folgt logisch

In other words: for each interpretation I ,
if $I \models \varphi$ for all $\varphi \in \Phi$, then also $I \models \psi$.

Question

How can we automatically compute if $\Phi \models \psi$?

- One possibility: Build a truth table. (How?)
- Are there “better” possibilities that (potentially) avoid generating the whole truth table?

Reasoning: Deduction Theorem

Proposition (deduction theorem)

Let Φ be a finite set of formulas and let ψ be a formula. Then

$$\Phi \models \psi \quad \text{iff} \quad \left(\bigwedge_{\varphi \in \Phi} \varphi \right) \rightarrow \psi \text{ is a tautology.}$$

German: Deduktionssatz

Reasoning: Deduction Theorem

Proposition (deduction theorem)

Let Φ be a finite set of formulas and let ψ be a formula. Then

$$\Phi \models \psi \quad \text{iff} \quad \left(\bigwedge_{\varphi \in \Phi} \varphi \right) \rightarrow \psi \text{ is a tautology.}$$

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Proof.

$$\Phi \models \psi$$

iff for each interpretation I : if $I \models \varphi$ for all $\varphi \in \Phi$, then $I \models \psi$

iff for each interpretation I : if $I \models \bigwedge_{\varphi \in \Phi} \varphi$, then $I \models \psi$

iff for each interpretation I : $I \not\models \bigwedge_{\varphi \in \Phi} \varphi$ or $I \models \psi$

iff for each interpretation I : $I \models (\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi$

iff $(\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi$ is tautology



Reasoning

Consequence of Deduction Theorem

Reasoning can be reduced to testing validity.

Algorithm

Question: Does $\Phi \models \psi$ hold?

- 1 test if $(\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi$ is tautology
- 2 if yes, then $\Phi \models \psi$, otherwise $\Phi \not\models \psi$

In the following: Can we test for validity “efficiently”,
i.e., without computing the whole truth table?

Resolution

Sets of Clauses

for the rest of this chapter:

- **prerequisite:** formulas in conjunctive normal form
- clause represented as a **set C of literals**
- formula represented as a **set Δ of clauses**

Example

Let $\varphi = (P \vee Q) \wedge \neg P$.

- φ in conjunctive normal form
- φ consists of clauses $(P \vee Q)$ and $\neg P$
- representation of φ as set of sets of literals: $\{\{P, Q\}, \{\neg P\}\}$

Distinguish $\square$ (empty clause) vs. $\emptyset$ (empty set of clauses).

Resolution: Idea

Observation

- Testing for validity can be reduced to testing unsatisfiability.
- formula φ valid iff $\neg\varphi$ unsatisfiable

Resolution: Idea

- method to test formula φ for unsatisfiability
- **idea**: derive new formulas from φ that logically follow from φ
- if empty clause $\square$ can be derived $\rightsquigarrow \varphi$ unsatisfiable

German: Resolution

The Resolution Rule

$$\frac{C_1 \cup \{\ell\}, C_2 \cup \{\bar{\ell}\}}{C_1 \cup C_2}$$

- “From $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$, we can conclude $C_1 \cup C_2$.”
- $C_1 \cup C_2$ is **resolvent** of **parent clauses** $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$.
- The literals ℓ and $\bar{\ell}$ are called **resolution literals**, the corresponding proposition is called **resolution variable**.
- resolvent follows logically from parent clauses (**Why?**)

German: Resolutionsregel, Resolvent, Elternklauseln, Resolutionslitterale, Resolutionsvariable

Example

- resolvent of $\{A, B, \neg C\}$ and $\{A, D, C\}$?
- resolvents of $\{\neg A, B, \neg C\}$ and $\{A, D, C\}$?

Resolution: Derivations

Definition (derivation)

Notation: $R(\Delta) = \Delta \cup \{C \mid C \text{ is resolvent of two clauses in } \Delta\}$

A clause D can be **derived** from Δ (in symbols $\Delta \vdash D$) if there is a sequence of clauses $C_1, \dots, C_n = D$ such that for all $i \in \{1, \dots, n\}$ we have $C_i \in R(\Delta \cup \{C_1, \dots, C_{i-1}\})$.

German: Ableitung, abgeleitet

Lemma (soundness of resolution)

If $\Delta \vdash D$, then $\Delta \models D$.

Does the converse direction hold as well (**completeness**)?

German: Korrektheit, Vollständigkeit

Resolution: Completeness?

The converse of the lemma does not hold in general.

example:

- $\{\{A, B\}, \{\neg B, C\}\} \models \{A, B, C\}$, but
- $\{\{A, B\}, \{\neg B, C\}\} \not\models \{A, B, C\}$

but: converse holds for special case empty clause $\square$ (no proof)

Theorem (refutation-completeness of resolution)

Δ is unsatisfiable iff $\Delta \vdash \square$

German: Widerlegungsvollständigkeit

consequences:

- Resolution is a complete proof method for testing unsatisfiability.
- Resolution can be used for general reasoning by reducing to a test for unsatisfiability.

Example

Let $\Phi = \{P \vee Q, \neg P\}$. Does $\Phi \models Q$ hold?

Solution

- test if $((P \vee Q) \wedge \neg P) \rightarrow Q$ is tautology
- equivalently: test if $((P \vee Q) \wedge \neg P) \wedge \neg Q$ is unsatisfiable
- resulting set of clauses: Φ' : $\{\{P, Q\}, \{\neg P\}, \{\neg Q\}\}$
- resolving $\{P, Q\}$ with $\{\neg P\}$ yields $\{Q\}$
- resolving $\{Q\}$ with $\{\neg Q\}$ yields $\square$
- observation: empty clause can be derived, hence Φ' unsatisfiable
- consequently $\Phi \models Q$

Resolution: Discussion

- Resolution is a complete proof method to test formulas for unsatisfiability.
- In the worst case, resolution proofs can take exponential time.
- In practice, a **strategy** which determines the next resolution step is needed.
- In the following chapter, we discuss the **DPLL** algorithm, which is a combination of backtracking and resolution.

Summary

Summary

- **Reasoning**: the formula ψ **follows from** the set of formulas Φ if all models of Φ are also models of ψ .
 - Reasoning can be reduced to testing validity (with the **deduction theorem**).
 - Testing validity can be reduced to testing unsatisfiability.
 - **Resolution** is a **refutation-complete** proof method applicable to formulas in conjunctive normal form.
- ~> can be used to test if a set of clauses is unsatisfiable