

Foundations of Artificial Intelligence

30. Propositional Logic: Reasoning and Resolution

Malte Helmert

University of Basel

April 25, 2022

Foundations of Artificial Intelligence

April 25, 2022 — 30. Propositional Logic: Reasoning and Resolution

30.1 Reasoning

30.2 Resolution

30.3 Summary

Propositional Logic: Overview

Chapter overview: propositional logic

- ▶ 29. Basics
- ▶ 30. Reasoning and Resolution
- ▶ 31. DPLL Algorithm
- ▶ 32. Local Search and Outlook

30.1 Reasoning

Reasoning: Intuition

Reasoning: Intuition

- ▶ Generally, formulas only represent an incomplete description of the world.
- ▶ In many cases, we want to know if a formula **logically follows** from (a set of) other formulas.
- ▶ What does this mean?

Reasoning: Intuition

- ▶ example: $\varphi = (P \vee Q) \wedge (R \vee \neg P) \wedge S$
- ▶ S holds in every model of φ .
What about P , Q and R ?
- ↪ consider all models of φ :

| P | Q | R | S |
|-----|-----|-----|-----|
| F | T | F | T |
| F | T | T | T |
| T | F | T | T |
| T | T | T | T |

Observation

- ▶ In all models of φ , the formula $Q \vee R$ holds as well.
- ▶ We say: " $Q \vee R$ **logically follows** from φ ."

Reasoning: Formally

Definition (logical consequence)

Let Φ be a set of formulas. A formula ψ **logically follows** from Φ (in symbols: $\Phi \models \psi$) if all models of Φ are also models of ψ .

German: logische Konsequenz, folgt logisch

In other words: for each interpretation I , if $I \models \varphi$ for all $\varphi \in \Phi$, then also $I \models \psi$.

Question

How can we automatically compute if $\Phi \models \psi$?

- ▶ One possibility: Build a truth table. (How?)
- ▶ Are there "better" possibilities that (potentially) avoid generating the whole truth table?

Reasoning: Deduction Theorem

Proposition (deduction theorem)

Let Φ be a finite set of formulas and let ψ be a formula. Then

$$\Phi \models \psi \quad \text{iff} \quad \left(\bigwedge_{\varphi \in \Phi} \varphi \right) \rightarrow \psi \text{ is a tautology.}$$

German: Deduktionsatz

Proof.

$\Phi \models \psi$
 iff for each interpretation I : if $I \models \varphi$ for all $\varphi \in \Phi$, then $I \models \psi$
 iff for each interpretation I : if $I \models \bigwedge_{\varphi \in \Phi} \varphi$, then $I \models \psi$
 iff for each interpretation I : $I \not\models \bigwedge_{\varphi \in \Phi} \varphi$ or $I \models \psi$
 iff for each interpretation I : $I \models (\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi$
 iff $(\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi$ is tautology □

Reasoning

Consequence of Deduction Theorem

Reasoning can be reduced to testing validity.

Algorithm

Question: Does $\Phi \models \psi$ hold?

- ① test if $(\bigwedge_{\varphi \in \Phi} \varphi) \rightarrow \psi$ is tautology
- ② if yes, then $\Phi \models \psi$, otherwise $\Phi \not\models \psi$

In the following: Can we test for validity “efficiently”, i.e., without computing the whole truth table?

30.2 Resolution

Sets of Clauses

for the rest of this chapter:

- ▶ **prerequisite:** formulas in conjunctive normal form
- ▶ clause represented as a **set C of literals**
- ▶ formula represented as a **set Δ of clauses**

Example

Let $\varphi = (P \vee Q) \wedge \neg P$.

- ▶ φ in conjunctive normal form
- ▶ φ consists of clauses $(P \vee Q)$ and $\neg P$
- ▶ representation of φ as set of sets of literals: $\{\{P, Q\}, \{\neg P\}\}$

Distinguish \square (empty clause) vs. \emptyset (empty set of clauses).

Resolution: Idea

Observation

- ▶ Testing for validity can be reduced to testing unsatisfiability.
- ▶ formula φ valid iff $\neg\varphi$ unsatisfiable

Resolution: Idea

- ▶ method to test formula φ for unsatisfiability
- ▶ **idea:** derive new formulas from φ that logically follow from φ
- ▶ if empty clause \square can be derived $\rightsquigarrow \varphi$ unsatisfiable

German: Resolution

The Resolution Rule

$$\frac{C_1 \cup \{\ell\}, C_2 \cup \{\bar{\ell}\}}{C_1 \cup C_2}$$

- ▶ “From $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$, we can conclude $C_1 \cup C_2$.”
- ▶ $C_1 \cup C_2$ is **resolvent** of **parent clauses** $C_1 \cup \{\ell\}$ and $C_2 \cup \{\bar{\ell}\}$.
- ▶ The literals ℓ and $\bar{\ell}$ are called **resolution literals**, the corresponding proposition is called **resolution variable**.
- ▶ resolvent follows logically from parent clauses (**Why?**)

German: Resolutionsregel, Resolvent, Elternklauseln, Resolutionslitterale, Resolutionsvariable

Example

- ▶ resolvent of $\{A, B, \neg C\}$ and $\{A, D, C\}$?
- ▶ resolvents of $\{\neg A, B, \neg C\}$ and $\{A, D, C\}$?

Resolution: Derivations

Definition (derivation)

Notation: $R(\Delta) = \Delta \cup \{C \mid C \text{ is resolvent of two clauses in } \Delta\}$

A clause D can be **derived** from Δ (in symbols $\Delta \vdash D$) if there is a sequence of clauses $C_1, \dots, C_n = D$ such that for all $i \in \{1, \dots, n\}$ we have $C_i \in R(\Delta \cup \{C_1, \dots, C_{i-1}\})$.

German: Ableitung, abgeleitet

Lemma (soundness of resolution)

If $\Delta \vdash D$, then $\Delta \models D$.

Does the converse direction hold as well (**completeness**)?

German: Korrektheit, Vollständigkeit

Resolution: Completeness?

The converse of the lemma does not hold in general.

example:

- ▶ $\{\{A, B\}, \{\neg B, C\}\} \models \{A, B, C\}$, but
- ▶ $\{\{A, B\}, \{\neg B, C\}\} \not\models \{A, B, C\}$

but: converse holds for special case of empty clause \square (no proof)

Theorem (refutation-completeness of resolution)

Δ is unsatisfiable iff $\Delta \vdash \square$

German: Widerlegungsvollständigkeit

consequences:

- ▶ Resolution is a complete proof method for testing unsatisfiability.
- ▶ Resolution can be used for general reasoning by reducing to a test for unsatisfiability.

Example

Let $\Phi = \{P \vee Q, \neg P\}$. Does $\Phi \models Q$ hold?

Solution

- ▶ test if $((P \vee Q) \wedge \neg P) \rightarrow Q$ is tautology
- ▶ equivalently: test if $((P \vee Q) \wedge \neg P) \wedge \neg Q$ is unsatisfiable
- ▶ resulting set of clauses: $\Phi': \{\{P, Q\}, \{\neg P\}, \{\neg Q\}\}$
- ▶ resolving $\{P, Q\}$ with $\{\neg P\}$ yields $\{Q\}$
- ▶ resolving $\{Q\}$ with $\{\neg Q\}$ yields \square
- ▶ observation: empty clause can be derived, hence Φ' unsatisfiable
- ▶ consequently $\Phi \models Q$

Resolution: Discussion

- ▶ Resolution is a complete proof method to test formulas for unsatisfiability.
- ▶ In the worst case, resolution proofs can take exponential time.
- ▶ In practice, a **strategy** which determines the next resolution step is needed.
- ▶ In the following chapter, we discuss the **DPLL** algorithm, which is a combination of backtracking and resolution.

30.3 Summary

Summary

- ▶ **Reasoning**: the formula ψ **follows from** the set of formulas Φ if all models of Φ are also models of ψ .
- ▶ Reasoning can be reduced to testing validity (with the **deduction theorem**).
- ▶ Testing validity can be reduced to testing unsatisfiability.
- ▶ **Resolution** is a **refutation-complete** proof method applicable to formulas in conjunctive normal form.
- ↔ can be used to test if a set of clauses is unsatisfiable