

Foundations of Artificial Intelligence

29. Propositional Logic: Basics

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Classification

classification:

Propositional Logic

environment:

- **static** vs. dynamic
- **deterministic** vs. non-deterministic vs. stochastic
- **fully** vs. partially vs. not **observable**
- **discrete** vs. continuous
- **single-agent** vs. multi-agent

problem solving method:

- problem-specific vs. **general** vs. learning

(applications also in more complex environments)

Propositional Logic: Overview

Chapter overview: propositional logic

- 29. Basics
- 30. Reasoning and Resolution
- 31. DPLL Algorithm
- 32. Local Search and Outlook

Motivation

Propositional Logic: Motivation

propositional logic

- modeling and representing problems and knowledge
- basics for **general** problem descriptions and solving strategies
($\rightsquigarrow$ **automated planning** $\rightsquigarrow$ later in this course)
- allows for automated **reasoning**

German: Aussagenlogik, automatisches Schliessen

Relationship to CSPs

- **previous topic:** constraint satisfaction problems
- satisfiability problem in propositional logic can be viewed as **non-binary CSP** over $\{\mathbf{F}, \mathbf{T}\}$
- formula encodes constraints
- solution: satisfying assignment of values to variables
- SAT algorithms for this problem: $\rightsquigarrow$ DPLL (**next week**)

Propositional Logic: Description of State Spaces

propositional variables for missionaries and cannibals problem:

two-missionaries-are-on-left-shore

one-cannibal-is-on-left-shore

boat-is-on-left-shore

...

- problem description for general problem solvers
- states represented as truth values of atomic **propositions**

German: Aussagenvariablen

Propositional Logic: Intuition

propositions: atomic statements over the world
that cannot be divided further

Propositions with **logical connectives** like
“and”, “or” and “not” form the propositional formulas.

German: logische Verknüpfungen

Syntax

Syntax

Σ alphabet of propositions

(e.g., $\{P, Q, R, \dots\}$ or $\{X_1, X_2, X_3, \dots\}$).

Definition (propositional formula)

- $\top$ and $\perp$ are formulas.
- Every proposition in Σ is an (atomic) formula.
- If φ is a formula, then $\neg\varphi$ is a formula (**negation**).
- If φ and ψ are formulas, then so are
 - $(\varphi \wedge \psi)$ (**conjunction**)
 - $(\varphi \vee \psi)$ (**disjunction**)
 - $(\varphi \rightarrow \psi)$ (**implication**)

German: aussagenlogische Formel, atomare Formel, Konjunktion, Disjunktion, Implikation

binding strength: $(\neg) > (\wedge) > (\vee) > (\rightarrow)$
(may omit redundant parentheses)

Semantics

Semantics

A formula is **true** or **false**,
depending on the **interpretation** of the propositions.

Semantics: Intuition

- A proposition p is either true or false.
The truth value of p is determined by an **interpretation**.
- The truth value of a formula follows from
the truth values of the propositions.

Example

$$\varphi = (P \vee Q) \wedge R$$

- If P and Q are false, then φ is false
(independent of the truth value of R).
- If P and R are true, then φ is true
(independent of the truth value of Q).

Semantics: Formally

- defined over **interpretation** $I : \Sigma \rightarrow \{\mathbf{T}, \mathbf{F}\}$
- interpretation I : **assignment** of propositions in Σ
- When is a formula φ true under interpretation I ?
symbolically: When does $I \models \varphi$ hold?

German: Interpretation, Belegung

Semantics: Formally

Definition ($I \models \varphi$)

- $I \models \top$ and $I \not\models \perp$
- $I \models P$ iff $I(P) = \mathbf{T}$ for $P \in \Sigma$
- $I \models \neg\varphi$ iff $I \not\models \varphi$
- $I \models (\varphi \wedge \psi)$ iff $I \models \varphi$ and $I \models \psi$
- $I \models (\varphi \vee \psi)$ iff $I \models \varphi$ or $I \models \psi$
- $I \models (\varphi \rightarrow \psi)$ iff $I \not\models \varphi$ or $I \models \psi$
- $I \models \Phi$ for a set of formulas Φ iff $I \models \varphi$ for all $\varphi \in \Phi$

German: I erfüllt φ , φ gilt unter I

Examples

Example (Interpretation I)

$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$

Which formulas are true under I ?

- $\varphi_1 = \neg(P \wedge Q) \wedge (R \wedge \neg S)$. Does $I \models \varphi_1$ hold?
- $\varphi_2 = (P \wedge Q) \wedge \neg(R \wedge \neg S)$. Does $I \models \varphi_2$ hold?
- $\varphi_3 = (R \rightarrow P)$. Does $I \models \varphi_3$ hold?

Terminology

Definition (model)

An interpretation I is called a **model** of φ if $I \models \varphi$.

German: Modell

Definition (satisfiable etc.)

A formula φ is called

- **satisfiable** if there is an interpretation I such that $I \models \varphi$.
- **unsatisfiable** if φ is not satisfiable.
- **falsifiable** if there is an interpretation I such that $I \not\models \varphi$.
- **valid** (= a **tautology**) if $I \models \varphi$ for all interpretations I .

German: erfüllbar, unerfüllbar, falsifizierbar,
allgemeingültig (gültig, Tautologie)

Terminology

Definition (logical equivalence)

Formulas φ and ψ are called **logically equivalent** ($\varphi \equiv \psi$) if for all interpretations I : $I \models \varphi$ iff $I \models \psi$.

German: logisch äquivalent

Truth Tables

Truth Tables

How to determine automatically if a given formula is (un)satisfiable, falsifiable, valid?

$\rightsquigarrow$ simple method: **truth tables**

example: Is $\varphi = ((P \vee H) \wedge \neg H) \rightarrow P$ valid?

P	H	$P \vee H$	$((P \vee H) \wedge \neg H)$	$((P \vee H) \wedge \neg H) \rightarrow P$
F	F	F	F	T
F	T	T	F	T
T	F	T	T	T
T	T	T	F	T

$I \models \varphi$ for all interpretations $I \rightsquigarrow \varphi$ is valid.

- satisfiability, falsifiability, unsatisfiability?

Normal Forms

Normal Forms: Terminology

Definition (literal)

If $P \in \Sigma$, then the formulas P and $\neg P$ are called **literals**.

P is called **positive literal**, $\neg P$ is called **negative literal**.

The **complementary literal** to P is $\neg P$ and vice versa.

For a literal ℓ , the complementary literal to ℓ is denoted with $\bar{\ell}$.

German: Literal, positives/negatives/komplementäres Literal

Question: What is the difference between $\bar{\ell}$ and $\neg\ell$?

Normal Forms: Terminology

Definition (clause)

A disjunction of 0 or more literals is called a **clause**.

The **empty clause** $\perp$ is also written as $\square$.

Clauses consisting of only one literal are called **unit clauses**.

German: Klausel

Definition (monomial)

A conjunction of 0 or more literals is called a **monomial**.

German: Monom

Normal Forms

Definition (normal forms)

A formula φ is in **conjunctive normal form** (CNF, clause form) if φ is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^n \left(\bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

A formula φ is in **disjunctive normal form** (DNF) if φ is a disjunction of 0 or more monomials:

$$\varphi = \bigvee_{i=1}^n \left(\bigwedge_{j=1}^{m_i} \ell_{i,j} \right)$$

German: konjunktive Normalform, disjunktive Normalform

Normal Forms

For every propositional formula, there exists a logically equivalent propositional formula in CNF and in DNF.

Conversion to CNF

important rules for conversion to CNF:

- $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$ (($\rightarrow$)-elimination)
- $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$ (De Morgan)
- $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$ (De Morgan)
- $\neg\neg\varphi \equiv \varphi$ (double negation)
- $((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta))$ (distributivity)

There are formulas φ for which every logically equivalent formula in CNF and DNF is exponentially longer than φ .

Summary

Summary (1)

- **Propositional logic** forms the basis for a general representation of problems and knowledge.
- **Propositions** (atomic formulas) are statements over the world which cannot be divided further.
- **Propositional formulas** combine atomic formulas with $\neg$, $\wedge$, $\vee$, $\rightarrow$ to more complex statements.
- **Interpretations** determine which atomic formulas are true and which ones are false.

Summary (2)

- important terminology:
 - model
 - satisfiable, unsatisfiable, falsifiable, valid
 - logically equivalent
- different kinds of formulas:
 - atomic formulas and literals
 - clauses and monomials
 - conjunctive normal form and disjunctive normal form