

# Foundations of Artificial Intelligence

## 29. Propositional Logic: Basics

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April 20, 2022 — 29. Propositional Logic: Basics

## 29.1 Motivation

## 29.2 Syntax

## 29.3 Semantics

## 29.4 Normal Forms

## 29.5 Summary

## Classification

classification:

Propositional Logic

environment:

- ▶ **static** vs. dynamic
- ▶ **deterministic** vs. non-deterministic vs. stochastic
- ▶ **fully** vs. partially vs. not **observable**
- ▶ **discrete** vs. continuous
- ▶ **single-agent** vs. multi-agent

problem solving method:

- ▶ problem-specific vs. **general** vs. learning

(applications also in more complex environments)

## Propositional Logic: Overview

Chapter overview: propositional logic

- ▶ **29. Basics**
- ▶ 30. Reasoning and Resolution
- ▶ 31. DPLL Algorithm
- ▶ 32. Local Search and Outlook

## 29.1 Motivation

## Propositional Logic: Motivation

### propositional logic

- ▶ modeling and representing problems and knowledge
- ▶ basics for **general** problem descriptions and solving strategies ( $\rightsquigarrow$  automated planning  $\rightsquigarrow$  later in this course)
- ▶ allows for automated **reasoning**

German: Aussagenlogik, automatisches Schliessen

## Relationship to CSPs

- ▶ **previous topic**: constraint satisfaction problems
- ▶ satisfiability problem in propositional logic can be viewed as **non-binary CSP** over  $\{\mathbf{F}, \mathbf{T}\}$
- ▶ formula encodes constraints
- ▶ solution: satisfying assignment of values to variables
- ▶ SAT algorithms for this problem:  $\rightsquigarrow$  DPLL (**next week**)

## Propositional Logic: Description of State Spaces

### propositional variables for missionaries and cannibals problem:

```
two-missionaries-are-on-left-shore
one-cannibal-is-on-left-shore
boat-is-on-left-shore
...
```

- ▶ problem description for general problem solvers
- ▶ states represented as truth values of atomic **propositions**

German: Aussagenvariablen

## Propositional Logic: Intuition

**propositions:** atomic statements over the world that cannot be divided further

Propositions with **logical connectives** like “and”, “or” and “not” form the propositional formulas.

**German:** logische Verknüpfungen

## 29.2 Syntax

## Syntax

$\Sigma$  alphabet of propositions  
(e.g.,  $\{P, Q, R, \dots\}$  or  $\{X_1, X_2, X_3, \dots\}$ ).

### Definition (propositional formula)

- ▶  $\top$  and  $\perp$  are formulas.
- ▶ Every proposition in  $\Sigma$  is an (atomic) formula.
- ▶ If  $\varphi$  is a formula, then  $\neg\varphi$  is a formula (**negation**).
- ▶ If  $\varphi$  and  $\psi$  are formulas, then so are
  - ▶  $(\varphi \wedge \psi)$  (**conjunction**)
  - ▶  $(\varphi \vee \psi)$  (**disjunction**)
  - ▶  $(\varphi \rightarrow \psi)$  (**implication**)

**German:** aussagenlogische Formel, atomare Formel, Konjunktion, Disjunktion, Implikation

**binding strength:**  $(\neg) > (\wedge) > (\vee) > (\rightarrow)$   
(may omit redundant parentheses)

## 29.3 Semantics

## Semantics

A formula is **true** or **false**,  
depending on the **interpretation** of the propositions.

### Semantics: Intuition

- ▶ A proposition  $p$  is either true or false.  
The truth value of  $p$  is determined by an **interpretation**.
- ▶ The truth value of a formula follows from the truth values of the propositions.

### Example

$$\varphi = (P \vee Q) \wedge R$$

- ▶ If  $P$  and  $Q$  are false, then  $\varphi$  is false  
(independent of the truth value of  $R$ ).
- ▶ If  $P$  and  $R$  are true, then  $\varphi$  is true  
(independent of the truth value of  $Q$ ).

## Semantics: Formally

- ▶ defined over **interpretation**  $I: \Sigma \rightarrow \{\mathbf{T}, \mathbf{F}\}$
- ▶ interpretation  $I$ : **assignment** of propositions in  $\Sigma$
- ▶ When is a formula  $\varphi$  true under interpretation  $I$ ?  
symbolically: When does  $I \models \varphi$  hold?

German: Interpretation, Belegung

## Semantics: Formally

### Definition ( $I \models \varphi$ )

- ▶  $I \models \mathbf{T}$  and  $I \not\models \perp$
- ▶  $I \models P$  iff  $I(P) = \mathbf{T}$  for  $P \in \Sigma$
- ▶  $I \models \neg\varphi$  iff  $I \not\models \varphi$
- ▶  $I \models (\varphi \wedge \psi)$  iff  $I \models \varphi$  and  $I \models \psi$
- ▶  $I \models (\varphi \vee \psi)$  iff  $I \models \varphi$  or  $I \models \psi$
- ▶  $I \models (\varphi \rightarrow \psi)$  iff  $I \not\models \varphi$  or  $I \models \psi$
- ▶  $I \models \Phi$  for a set of formulas  $\Phi$  iff  $I \models \varphi$  for all  $\varphi \in \Phi$

German:  $I$  erfüllt  $\varphi$ ,  $\varphi$  gilt unter  $I$

## Examples

### Example (Interpretation $I$ )

$$I = \{P \mapsto \mathbf{T}, Q \mapsto \mathbf{T}, R \mapsto \mathbf{F}, S \mapsto \mathbf{F}\}$$

### Which formulas are true under $I$ ?

- ▶  $\varphi_1 = \neg(P \wedge Q) \wedge (R \wedge \neg S)$ . Does  $I \models \varphi_1$  hold?
- ▶  $\varphi_2 = (P \wedge Q) \wedge \neg(R \wedge \neg S)$ . Does  $I \models \varphi_2$  hold?
- ▶  $\varphi_3 = (R \rightarrow P)$ . Does  $I \models \varphi_3$  hold?

## Terminology

### Definition (model)

An interpretation  $I$  is called a **model** of  $\varphi$  if  $I \models \varphi$ .

German: Modell

### Definition (satisfiable etc.)

A formula  $\varphi$  is called

- ▶ **satisfiable** if there is an interpretation  $I$  such that  $I \models \varphi$ .
- ▶ **unsatisfiable** if  $\varphi$  is not satisfiable.
- ▶ **falsifiable** if there is an interpretation  $I$  such that  $I \not\models \varphi$ .
- ▶ **valid** (= a **tautology**) if  $I \models \varphi$  for all interpretations  $I$ .

German: erfüllbar, unerfüllbar, falsifizierbar, allgemeingültig (gültig, Tautologie)

## Terminology

### Definition (logical equivalence)

Formulas  $\varphi$  and  $\psi$  are called **logically equivalent** ( $\varphi \equiv \psi$ ) if for all interpretations  $I$ :  $I \models \varphi$  iff  $I \models \psi$ .

German: logisch äquivalent

## Truth Tables

### Truth Tables

How to determine automatically if a given formula is (un)satisfiable, falsifiable, valid?

↔ simple method: **truth tables**

example: Is  $\varphi = ((P \vee H) \wedge \neg H) \rightarrow P$  valid?

$P$	$H$	$P \vee H$	$((P \vee H) \wedge \neg H)$	$((P \vee H) \wedge \neg H) \rightarrow P$
F	F	F	F	T
F	T	T	F	T
T	F	T	T	T
T	T	T	F	T

$I \models \varphi$  for all interpretations  $I$  ↔  $\varphi$  is valid.

- ▶ satisfiability, falsifiability, unsatisfiability?

## 29.4 Normal Forms

## Normal Forms: Terminology

### Definition (literal)

If  $P \in \Sigma$ , then the formulas  $P$  and  $\neg P$  are called **literals**.

$P$  is called **positive literal**,  $\neg P$  is called **negative literal**.

The **complementary literal** to  $P$  is  $\neg P$  and vice versa.

For a literal  $\ell$ , the complementary literal to  $\ell$  is denoted with  $\bar{\ell}$ .

**German:** Literal, positives/negatives/komplementäres Literal

**Question:** What is the difference between  $\bar{\ell}$  and  $\neg\ell$ ?

## Normal Forms: Terminology

### Definition (clause)

A disjunction of 0 or more literals is called a **clause**.

The **empty clause**  $\perp$  is also written as  $\square$ .

Clauses consisting of only one literal are called **unit clauses**.

**German:** Klausel

### Definition (monomial)

A conjunction of 0 or more literals is called a **monomial**.

**German:** Monom

## Normal Forms

### Definition (normal forms)

A formula  $\varphi$  is in **conjunctive normal form** (CNF, clause form)

if  $\varphi$  is a conjunction of 0 or more clauses:

$$\varphi = \bigwedge_{i=1}^n \left( \bigvee_{j=1}^{m_i} \ell_{i,j} \right)$$

A formula  $\varphi$  is in **disjunctive normal form** (DNF)

if  $\varphi$  is a disjunction of 0 or more monomials:

$$\varphi = \bigvee_{i=1}^n \left( \bigwedge_{j=1}^{m_i} \ell_{i,j} \right)$$

**German:** konjunktive Normalform, disjunktive Normalform

## Normal Forms

For every propositional formula, there exists a logically equivalent propositional formula in CNF and in DNF.

### Conversion to CNF

important rules for conversion to CNF:

- ▶  $(\varphi \rightarrow \psi) \equiv (\neg\varphi \vee \psi)$  (( $\rightarrow$ )-elimination)
- ▶  $\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$  (De Morgan)
- ▶  $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$  (De Morgan)
- ▶  $\neg\neg\varphi \equiv \varphi$  (double negation)
- ▶  $((\varphi \wedge \psi) \vee \eta) \equiv ((\varphi \vee \eta) \wedge (\psi \vee \eta))$  (distributivity)

There are formulas  $\varphi$  for which every logically equivalent formula in CNF and DNF is exponentially longer than  $\varphi$ .

## 29.5 Summary

## Summary (1)

- ▶ **Propositional logic** forms the basis for a general representation of problems and knowledge.
- ▶ **Propositions** (atomic formulas) are statements over the world which cannot be divided further.
- ▶ **Propositional formulas** combine atomic formulas with  $\neg$ ,  $\wedge$ ,  $\vee$ ,  $\rightarrow$  to more complex statements.
- ▶ **Interpretations** determine which atomic formulas are true and which ones are false.

## Summary (2)

- ▶ important terminology:
  - ▶ **model**
  - ▶ **satisfiable, unsatisfiable, falsifiable, valid**
  - ▶ **logically equivalent**
- ▶ different kinds of formulas:
  - ▶ **atomic formulas and literals**
  - ▶ **clauses and monomials**
  - ▶ **conjunctive normal form and disjunctive normal form**