Foundations of Artificial Intelligence

22. Constraint Satisfaction Problems: Introduction and Examples

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22.1 Introduction

22.2 Examples

22.3 Summary

Classification

Classification:

Constraint Satisfaction Problems
environment:

- ▶ **static** vs. dynamic
- ▶ **deterministic** vs. non-deterministic vs. stochastic
- ▶ **fully** vs. partially vs. not **observable**
- ▶ **discrete** vs. continuous
- ▶ **single-agent** vs. multi-agent

problem solving method:

- ▶ problem-specific vs. **general** vs. learning

Constraint Satisfaction Problems: Overview

Chapter overview: constraint satisfaction problems

- ▶ 22.–23. Introduction
  - ▶ 22. Introduction and Examples
  - ▶ 23. Constraint Networks
- ▶ 24.–26. Basic Algorithms
- ▶ 27.–28. Problem Structure

22.1 Introduction

Constraints

> **What is a Constraint?**
> a condition that every solution to a problem must satisfy

German: Einschränkung, Nebenbedingung (math.)

Examples: Where do constraints occur?

▶ mathematics: requirements on solutions of optimization problems (e.g., equations, inequalities)

▶ software testing: specification of invariants to check data consistency (e.g., assertions)

▶ databases: integrity constraints

Constraint Satisfaction Problems: Informally

Given:

▶ set of variables with corresponding domains

▶ set of constraints that the variables must satisfy

▶ most commonly binary, i.e., every constraint refers to two variables

Solution:

▶ assignment to the variables that satisfies all constraints

German: Variablen, Constraints, binär, Belegung

22.2 Examples

Examples

> ### Examples
> - ► 8 queens problem
> - ► Latin squares
> - ► Sudoku
> - ► graph coloring
> - ► satisfiability in propositional logic

German: 8-Damen-Problem, lateinische Quadrate, Sudoku, Graphfärbung, Erfüllbarkeitsproblem der Aussagenlogik

more complex examples:
- ► systems of equations and inequalities
- ► database queries

Example: 8 Queens Problem (Reminder)

(reminder from previous two chapters)

> ### 8 Queens Problem
> How can we
> - ► place 8 queens on a chess board
> - ► such that no two queens threaten each other?

- ► originally proposed in 1848
- ► variants: board size; other pieces; higher dimension

There are 92 solutions, or 12 solutions if we do not count symmetric solutions (under rotation or reflection) as distinct.

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8 Queens Problem: Example Solution
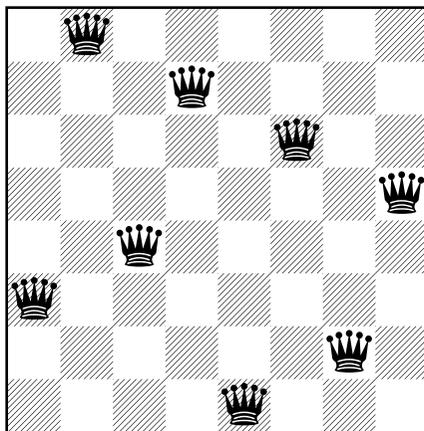
example solution for the 8 queens problem

Example: Latin Squares

> ### Latin Squares
> How can we
> - ► build an $n \times n$ matrix with n symbols
> - ► such that every symbol occurs exactly once in every row and every column?

$$[1] \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

There exist 12 different Latin squares of size 3, 576 of size 4, 161 280 of size 5, ..., 5 524 751 496 156 892 842 531 225 600 of size 9.

Example: Sudoku

Sudoku

How can we

- completely fill an already partially filled 9×9 matrix with numbers between 1–9
- such that each row, each column, and each of the nine 3×3 blocks contains every number exactly once?

relationship to Latin squares?

Sudoku: Trivia

- well-formed Sudokus have **exactly one** solution
- to achieve well-formedness, ≥ 17 cells must be filled already (McGuire et al., 2012)
- 6 670 903 752 021 072 936 960 solutions
- only 5 472 730 538 "non-symmetrical" solutions

Example: Graph Coloring

Graph Coloring

How can we

- color the vertices of a given graph using k colors
- such that two neighboring vertices never have the same color?

(The graph and k are problem parameters.)

NP-complete problem

- even for the special case of planar graphs and $k = 3$
- easy for $k = 2$ (also for general graphs)

Relationship to Sudoku?

Four Color Problem

famous problem in mathematics: **Four Color Problem**

- Is it always possible to color a planar graph with 4 colors?
- conjectured by Francis Guthrie (1852)
- 1890 first proof that 5 colors suffice
- several wrong proofs surviving for over 10 years
- solved by Appel and Haken in 1976: 4 colors suffice
- Appel and Haken reduced the problem to 1936 cases, which were then checked by computers
- first famous mathematical problem solved (partially) by computers
  $\rightsquigarrow$ led to controversy: is this a mathematical proof?

Numberphile video:
`https://www.youtube.com/watch?v=NgbK43jB4rQ`

Satisfiability in Propositional Logic

> **Satisfiability in Propositional Logic**
> How can we
> - assign truth values (true/false) to a set of propositional variables
> - such that a given set of clauses
>   (formulas of the form $X \vee \neg Y \vee Z$) is satisfied (true)?

remarks:

- NP-complete (Cook 1971; Levin 1973)
- formulas expressed as clauses (instead of arbitrary propositional formulas) is no restriction
- clause length bounded by 3 would not be a restriction

relationship to previous problems (e.g., Sudoku)?

Practical Applications

- There are thousands of practical applications of constraint satisfaction problems.
- This statement is true already for the satisfiability problem of propositional logic.

some examples:

- verification of hardware and software
- timetabling (e.g., generating time schedules, room assignments for university courses)
- assignment of frequency spectra
  (e.g., broadcasting, mobile phones)

22.3 Summary

Summary

- constraint satisfaction:
  - find assignment for a set of variables
  - with given variable domains
  - that satisfies a given set of constraints.
- examples:
  - 8 queens problem
  - Latin squares
  - Sudoku
  - graph coloring
  - satisfiability in propositional logic
  - many practical applications