Theory of Computer Science

E3. Cook-Levin Theorem

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Overview: Course

contents of this course:

- logic ✓
  - ▷ How can knowledge be represented? How can reasoning be automated?
- automata theory and formal languages ✓
  - ▷ What is a computation?
- computability theory ✓
  - ▷ What can be computed at all?
- complexity theory
  - ▷ What can be computed efficiently?

Overview: Complexity Theory

Complexity Theory

E1. Motivation and Introduction

E2. P, NP and Polynomial Reductions

E3. Cook-Levin Theorem

E4. Some NP-Complete Problems, Part I

E5. Some NP-Complete Problems, Part II

Further Reading (German)

Literature for this Chapter (German)

Theoretische Informatik – kurz gefasst
by Uwe Schöning (5th edition)

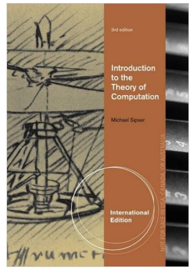
- Chapter 3.2

Further Reading (English)

Literature for this Chapter (English)

Introduction to the Theory of Computation
by Michael Sipser (3rd edition)

- Chapter 7.4



E3.1 Cook-Levin Theorem

SAT is NP-complete

Definition (SAT)

The problem SAT (satisfiability) is defined as follows:

Given: a propositional logic formula φ

Question: Is φ satisfiable?

Theorem (Cook, 1971; Levin, 1973)

SAT *is NP-complete.*

Proof.

SAT $\in$ NP: guess and check.
SAT is NP-hard: somewhat more complicated (to be continued)

...

NP-hardness of SAT (1)

Proof (continued).

We must show: $A \leq_p \mathrm{SAT}$ for all $A \in \mathrm{NP}$.

Let A be an arbitrary problem in NP.

We have to find a polynomial reduction of A to SAT, i. e., a function f computable in polynomial time such that for every input word w over the alphabet of A:

$w \in A$ iff $f(w)$ is a satisfiable propositional formula. ...

NP-hardness of SAT (2)

Proof (continued).

Because $A \in \mathrm{NP}$, there is an NTM M and a polynomial p such that M accepts the problem A in time p.

Idea: construct a formula that encodes the possible configurations which M can reach in time $p(|w|)$ on input w and that is satisfiable if and only if an end configuration can be reached in this time. ...

NP-hardness of SAT (3)

Proof (continued).

Let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, \square, E\rangle$ be an NTM for A, and let p be a polynomial bounding the computation time of M. Without loss of generality, $p(n) \geq n$ for all n.

Let $w = w_1 \dots w_n \in \Sigma^*$ be the input for M.

We number the tape positions with integers (positive and negative) such that the TM head initially is on position 1.

Observation: within $p(n)$ computation steps the TM head can only reach positions in the set $\mathit{Pos} = \{-p(n)+1, -p(n)+2, \dots, -1, 0, 1, \dots, p(n)+1\}$.

Instead of infinitely many tape positions, we now only need to consider these (polynomially many!) positions. ...

NP-hardness of SAT (4)

Proof (continued).

We can encode configurations of M by specifying:

- what the current state of M is
- on which position in Pos the TM head is located
- which symbols from Γ the tape contains at positions Pos

$\rightsquigarrow$ can be encoded by propositional variables

To encode a full computation (rather than just one configuration), we need copies of these variables for each computation step.

We only need to consider the computation steps $\mathit{Steps} = \{0, 1, \dots, p(n)\}$ because M should accept within $p(n)$ steps. ...

NP-hardness of SAT (5)

Proof (continued).

Use the following propositional variables in formula $f(w)$:

- $state_{t,q}$ ($t \in Steps$, $q \in Q$)
  $\rightsquigarrow$ encodes the state of the NTM in the t-th configuration
- $head_{t,i}$ ($t \in Steps$, $i \in Pos$)
  $\rightsquigarrow$ encodes the head position in the t-th configuration
- $tape_{t,i,a}$ ($t \in Steps$, $i \in Pos$, $a \in \Gamma$)
  $\rightsquigarrow$ encodes the tape content in the t-th configuration

Construct $f(w)$ such that every satisfying interpretation

- describes a sequence of TM configurations
- that begins with the start configuration,
- reaches an accepting configuration
- and follows the TM rules in δ

...

NP-hardness of SAT (6)

Proof (continued).

Auxiliary formula:

$$oneof\ X := \left(\bigvee_{x \in X} x\right) \wedge \neg \left(\bigvee_{x \in X} \bigvee_{y \in X \setminus \{x\}} (x \wedge y)\right)$$

Auxiliary notation:

The symbol $\perp$ stands for an arbitrary unsatisfiable formula (e.g., $(A \wedge \neg A)$, where A is an arbitrary proposition). ...

NP-hardness of SAT (7)

Proof (continued).

1. describe the configurations of the TM:

$$Valid := \bigwedge_{t \in Steps} \Big(oneof\ \{state_{t,q} \mid q \in Q\} \wedge oneof\ \{head_{t,i} \mid i \in Pos\} \wedge \bigwedge_{i \in Pos} oneof\ \{tape_{t,i,a} \mid a \in \Gamma\} \Big)$$

...

NP-hardness of SAT (8)

Proof (continued).

2. begin in the start configuration

$$Init := state_{0,q_0} \wedge \mathrm{head}_{0,1} \wedge \bigwedge_{i=1}^{n} tape_{0,i,w_i} \wedge \bigwedge_{i \in Pos \setminus \{1,\dots,n\}} tape_{0,i,\square}$$

...

NP-hardness of SAT (9)

Proof (continued).

3. reach an accepting configuration

$$Accept := \bigvee_{t \in Steps} \bigvee_{q_e \in E} state_{t,q_e}$$

...

NP-hardness of SAT (10)

Proof (continued).

4. follow the rules in δ:

$$Trans := \bigwedge_{t \in Steps} \left(\bigvee_{q_e \in E} state_{t,q_e} \vee \bigvee_{R \in \delta} Rule_{t,R} \right)$$

where. . . ...

NP-hardness of SAT (11)

Proof (continued).

4. follow the rules in δ (continued):

$$Rule_{t,\langle\langle q,a\rangle,\langle q',a',D\rangle\rangle} :=$$
$$state_{t,q} \wedge state_{t+1,q'} \wedge$$
$$\bigwedge_{i \in Pos} \left(head_{t,i} \rightarrow tape_{t,i,a} \wedge head_{t+1,i+D} \wedge tape_{t+1,i,a'} \right) \wedge$$
$$\bigwedge_{i \in Pos} \bigwedge_{a'' \in \Gamma} \left(\neg head_{t,i} \wedge tape_{t,i,a''} \rightarrow tape_{t+1,i,a''} \right)$$

- For D, interpret L $\rightsquigarrow -1$, N $\rightsquigarrow 0$, R $\rightsquigarrow +1$.
- special case: *tape* and *head* variables with a tape index $i + D$ outside of *Pos* are replaced by $\perp$; likewise all variables with a time index outside of *Steps*.

NP-hardness of SAT (12)

Proof (continued).

Putting the pieces together:

Set $f(w) := Valid \wedge Init \wedge Accept \wedge Trans$.

- $f(w)$ can be constructed in time polynomial in $|w|$.
- $w \in A$ iff M accepts w in $p(|w|)$ steps
  iff $f(w)$ is satisfiable
  iff $f(w) \in \mathrm{SAT}$

$\rightsquigarrow$ $A \leq_p \mathrm{SAT}$

Since $A \in$ NP was arbitrary, this is true for every $A \in$ NP.
Hence SAT is NP-hard and thus also NP-complete. □

E3.2 Summary

Summary

- The satisfiability problem of propositional logic (SAT) is NP-complete.
- Proof idea for NP-hardness:
  - Every problem in NP can be solved by an NTM in polynomial time $p(|w|)$ for input w.
  - Given a word w, construct a propositional logic formula φ that encodes the computation steps of the NTM on input w.
  - Construct φ so that it is satisfiable if and only if there is an accepting computation of length $p(|w|)$.