

Theory of Computer Science

E2. P, NP and Polynomial Reductions

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Overview: Course

contents of this course:

- logic ✓
 - ▷ How can knowledge be represented?
 - ▷ How can reasoning be automated?
- automata theory and formal languages ✓
 - ▷ What is a computation?
- computability theory ✓
 - ▷ What can be computed at all?
- complexity theory
 - ▷ What can be computed efficiently?

Overview: Complexity Theory

Complexity Theory

- E1. Motivation and Introduction
- E2. P, NP and Polynomial Reductions
- E3. Cook-Levin Theorem
- E4. Some NP-Complete Problems, Part I
- E5. Some NP-Complete Problems, Part II

Further Reading (German)

Literature for this Chapter (German)

Theoretische Informatik – kurz gefasst
by Uwe Schöning (5th edition)

- **Chapters 3.1 and 3.2**

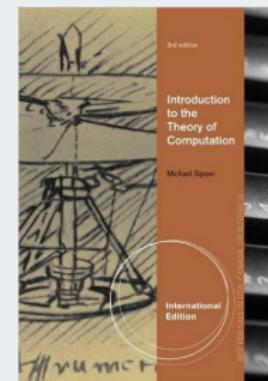


Further Reading (English)

Literature for this Chapter (English)

Introduction to the Theory of Computation
by Michael Sipser (3rd edition)

- **Chapter 7.1–7.4**



P and NP

Accepting a Word in Time n

Definition (Accepting a Word in Time n)

Let M be a DTM or NTM with input alphabet Σ , $w \in \Sigma^*$ a word and $n \in \mathbb{N}_0$.

M accepts w in time n if there is a sequence of configurations c_0, \dots, c_k with $k \leq n$, where:

- c_0 is the start configuration for w ,
- $c_0 \vdash c_1 \vdash \dots \vdash c_k$, and
- c_k is an end configuration.

German: M akzeptiert w in Zeit n

Accepting a Language in Time f

Definition (Accepting a Language in Time f)

Let M be a DTM or NTM with input alphabet Σ ,
 $L \subseteq \Sigma^*$ a language and $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ a function.

M accepts L in time f if:

- 1 for all words $w \in L$: M accepts w in time $f(|w|)$
- 2 for all words $w \notin L$: M does not accept w

German: M akzeptiert L in Zeit f

P and NP

Definition (P and NP)

P is the set of all languages L for which a **DTM** M and a **polynomial** p exist such that M accepts L in time p .

NP is the set of all languages L for which an **NTM** M and a **polynomial** p exist such that M accepts L in time p .

P and NP: Remarks

- Sets of languages like P and NP that are defined in terms of computation time of TMs (or other computation models) are called **complexity classes**.
- We know that $P \subseteq NP$. ([Why?](#))
- Whether the converse is also true is an open question: this is the famous **P-NP problem**.

[German](#): Komplexitätsklassen, P-NP-Problem

Example: $\text{DIRHAMILTONCYCLE} \in \text{NP}$

Example ($\text{DIRHAMILTONCYCLE} \in \text{NP}$)

The nondeterministic algorithm of Chapter E1 solves the problem and can be implemented on an NTM in polynomial time.

- Is $\text{DIRHAMILTONCYCLE} \in \text{P}$ also true?
- The answer is unknown.
- So far, only exponential deterministic algorithms for the problem are known.

Simulation of NTMs with DTMs

- Unlike DTM s , NTM s are not a **realistic** computation model: they cannot be directly implemented on computers.
- But NTM s can be **simulated** by systematically trying all computation paths, e. g., with a **breadth-first search**.

Simulation of NTMs with DTMs

- Unlike DTMs, NTMs are not a **realistic** computation model: they cannot be directly implemented on computers.
- But NTMs can be **simulated** by systematically trying all computation paths, e. g., with a **breadth-first search**.

More specifically:

- Let M be an NTM that accepts language L in time f , where $f(n) \geq n$ for all $n \in \mathbb{N}_0$.
- Then we can specify a DTM M' that accepts L in time f' , where $f'(n) = 2^{O(f(n))}$.
- **without proof** (cf. Sipser, Theorem 7.11)

Questions



Questions?

Polynomial Reductions

Polynomial Reductions: Idea

- **Reductions** are a common and powerful concept in computer science. We know them from Part D.
- The basic idea is that we solve a new problem by **reducing** it to a known problem.

Polynomial Reductions: Idea

- **Reductions** are a common and powerful concept in computer science. We know them from Part D.
- The basic idea is that we solve a new problem by **reducing** it to a known problem.
- In complexity theory we want to use reductions that allow us to prove statements of the following kind:

*Problem A can be solved efficiently
if problem B can be solved efficiently.*

- For this, we need a reduction from A to B that can be computed efficiently itself (otherwise it would be useless for efficiently solving A).

Polynomial Reductions

Definition (Polynomial Reduction)

Let $A \subseteq \Sigma^*$ and $B \subseteq \Gamma^*$ be decision problems.

We say that A can be polynomially reduced to B , written $A \leq_p B$, if there is a function $f : \Sigma^* \rightarrow \Gamma^*$ such that:

- f can be computed in polynomial time by a DTM
 - i. e., there is a polynomial p and a DTM M such that M computes $f(w)$ in at most $p(|w|)$ steps given input $w \in \Sigma^*$
- f reduces A to B
 - i. e., for all $w \in \Sigma^*$: $w \in A$ iff $f(w) \in B$

f is called a polynomial reduction from A to B

German: A polynomiell auf B reduzierbar,
polynomielle Reduktion von A auf B

Polynomial Reductions: Remarks

- Polynomial reductions are also called **Karp reductions** (after Richard Karp, who wrote a famous paper describing many such reductions in 1972).
- In practice, of course we do not have to specify a DTM for f : it just has to be clear that f can be computed in **polynomial time** by a **deterministic algorithm**.

Polynomial Reductions: Example (1)

Definition (HAMILTONCYCLE)

HAMILTONCYCLE is the following decision problem:

- **Given:** undirected graph $G = \langle V, E \rangle$
- **Question:** Does G contain a Hamilton cycle?

Polynomial Reductions: Example (2)

Definition (TSP)

TSP (traveling salesperson problem) is the following decision problem:

- **Given:** finite set $S \neq \emptyset$ of cities, symmetric cost function $cost : S \times S \rightarrow \mathbb{N}_0$, cost bound $K \in \mathbb{N}_0$
- **Question:** Is there a tour with total cost at most K , i. e., a permutation $\langle s_1, \dots, s_n \rangle$ of the cities with $\sum_{i=1}^{n-1} cost(s_i, s_{i+1}) + cost(s_n, s_1) \leq K$?

German: Problem der/des Handlungsreisenden

Polynomial Reductions: Example (3)

Theorem ($\text{HAMILTONCYCLE} \leq_p \text{TSP}$)

$\text{HAMILTONCYCLE} \leq_p \text{TSP}.$

Proof.

↔ blackboard



Properties of Polynomial Reductions (1)

Theorem (Properties of Polynomial Reductions)

Let A , B and C decision problems.

- ① If $A \leq_p B$ and $B \in P$, then $A \in P$.
- ② If $A \leq_p B$ and $B \in NP$, then $A \in NP$.
- ③ If $A \leq_p B$ and $A \notin P$, then $B \notin P$.
- ④ If $A \leq_p B$ and $A \notin NP$, then $B \notin NP$.
- ⑤ If $A \leq_p B$ and $B \leq_p C$, then $A \leq_p C$.

Properties of Polynomial Reductions (2)

Proof.

for 1.:

We must show that there is a DTM accepting A in polynomial time.

We know:

- There is a DTM M_B that accepts B in time p , where p is a polynomial.
- There is a DTM M_f that computes a reduction from A to B in time q , where q is a polynomial.

...

Properties of Polynomial Reductions (3)

Proof (continued).

Consider the machine M that first behaves like M_f , and then (after M_f stops) behaves like M_B on the output of M_f .

M accepts A :

- M behaves on input w as M_B does on input $f(w)$, so it accepts w if and only if $f(w) \in B$.
- Because f is a reduction, $w \in A$ iff $f(w) \in B$.

...

Properties of Polynomial Reductions (4)

Proof (continued).

Computation time of M on input w :

- first M_f runs on input w : $\leq q(|w|)$ steps
- then M_B runs on input $f(w)$: $\leq p(|f(w)|)$ steps
- $|f(w)| \leq |w| + q(|w|)$ because in $q(|w|)$ steps, M_f can write at most $q(|w|)$ additional symbols onto the tape

↝ total computation time $\leq q(|w|) + p(|f(w)|)$
 $\leq q(|w|) + p(|w| + q(|w|))$

↝ this is polynomial in $|w|$ ↝ $A \in P$.

...

Properties of Polynomial Reductions (5)

Proof (continued).

for 2.:

analogous to 1., only that M_B and M are NTMs

Properties of Polynomial Reductions (5)

Proof (continued).

for 2.:

analogous to 1., only that M_B and M are NTMs

of 3.+4.:

equivalent formulations of 1.+2. (contraposition)

Properties of Polynomial Reductions (5)

Proof (continued).

for 2.:

analogous to 1., only that M_B and M are NTMs

of 3.+4.:

equivalent formulations of 1.+2. (contraposition)

of 5.:

Let $A \leq_p B$ with reduction f and $B \leq_p C$ with reduction g .

Then $g \circ f$ is a reduction of A to C .

The computation time of the two computations in sequence
is polynomial by the same argument used in the proof for 1. □

Questions



Questions?

NP-Hardness and NP-Completeness

NP-Hardness and NP-Completeness

Definition (NP-Hard, NP-Complete)

Let B be a decision problem.

B is called **NP-hard** if $A \leq_p B$ for **all** problems $A \in \text{NP}$.

B is called **NP-complete** if $B \in \text{NP}$ and B is NP-hard.

German: NP-hart (selten: NP-schwierig), NP-vollständig

NP-Complete Problems: Meaning

- NP-hard problems are “at least as difficult” as all problems in NP.
- NP-complete problems are “the most difficult” problems in NP: **all** problems in NP can be reduced to them.

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- If $A \in P$ for **any** NP-complete problem, then $P = NP$. ([Why?](#))
- That means that either there are efficient algorithms for **all** NP-complete problems or for **none** of them.

NP-Complete Problems: Meaning

- NP-hard problems are “at least as difficult” as all problems in NP.
- NP-complete problems are “the most difficult” problems in NP: **all** problems in NP can be reduced to them.
- If $A \in P$ for **any** NP-complete problem, then $P = NP$. ([Why?](#))
- That means that either there are efficient algorithms for **all** NP-complete problems or for **none** of them.
- **Do NP-complete problems actually exist?**

Questions



Questions?

Summary

Summary

- P: languages accepted by **DTMs** in polynomial time
- NP: languages accepted by **NTMs** in polynomial time
- **polynomial reductions**: $A \leq_p B$ if
there is a total function f computable **in polynomial time**,
such that for all words w : $w \in A$ iff $f(w) \in B$
- $A \leq_p B$ implies that A is **“at most as difficult”** as B
- polynomial reductions are **transitive**
- **NP-hard** problems B : $A \leq_p B$ for **all** $A \in \text{NP}$
- **NP-complete** problems B : $B \in \text{NP}$ and B is NP-hard