

Theory of Computer Science

B1. Propositional Logic I

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February 27, 2017

Motivation

Exercise from Last Lecture

What's the secret of your long life?



I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Simplify this advice!

Why Logic?

- formalizing mathematics
 - What is a true statement?
 - What is a valid proof?
- basis of many tools in computer science
 - design of digital circuits
 - meaning of programming languages
 - semantics of databases; query optimization
 - verification of safety-critical hardware/software
 - knowledge representation in artificial intelligence
 - ...

Example: Group Theory

Example of a **group** (in mathematics): $\langle \mathbb{Z}, + \rangle$

- the set of integers with the addition operation

A **group** in general: $\langle G, \circ \rangle$

- G is a set and $\circ : G \times G \rightarrow G$ is called the **group operation**; we write “ $x \circ y$ ” instead of “ $\circ(x, y)$ ” (**infix** notation)

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For $\langle G, \circ \rangle$ to be a group, it must satisfy the **group axioms**:

- (G1) For all $x, y, z \in G$, $(x \circ y) \circ z = x \circ (y \circ z)$.
- There exists $e \in G$ (called the **neutral element**) such that:
 - (G2) for all $x \in G$, $x \circ e = x$, and
 - (G3) for all $x \in G$, there is a $y \in G$ with $x \circ y = e$.

German: Gruppe, Verknüpfung, Infix, Gruppenaxiome, neutrales Element

Example: Group Theory

Theorem (Existence of a left inverse)

Let $\langle G, \circ \rangle$ be a group with neutral element e .

For all $x \in G$ there is a $y \in G$ with $y \circ x = e$.

Proof.

Consider an arbitrary $x \in G$.

Because of G3, there is a y with $x \circ y = e$ (*).

Also because of G3, for this y there is a z with $y \circ z = e$ (**).

It follows that:

$$\begin{aligned}
 y \circ x &\stackrel{(G2)}{=} (y \circ x) \circ e \stackrel{(**)}{=} (y \circ x) \circ (y \circ z) \\
 &\stackrel{(G1)}{=} y \circ (x \circ (y \circ z)) \stackrel{(G1)}{=} y \circ ((x \circ y) \circ z) \\
 &\stackrel{(*)}{=} y \circ (e \circ z) \stackrel{(G1)}{=} (y \circ e) \circ z \\
 &\stackrel{(G2)}{=} y \circ z \stackrel{(**)}{=} e
 \end{aligned}$$



What Logic is About

General Question:

- Given a set of axioms (e. g., group axioms)
- what can we **derive** from them?
(e. g., theorem about the existence of a left inverse)
- And on what basis may we argue?
(e. g., why does $y \circ x = (y \circ x) \circ e$ follow from axiom G2?)

↪ **logic**

Goal: “mechanical” proofs

- formal “game with letters”
- detached from a concrete meaning

Propositional Logic

Propositional logic is a simple logic without numbers or objects.

Building blocks of propositional logic:

- **propositions** are statements that can be either true or false
- **atomic propositions** cannot be split into sub-propositions
- **logical connectives** connect propositions to form new ones

German: Aussagenlogik, Aussage, atomare Aussage, Junktoren

Examples for Building Blocks



If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

- Every sentence is a proposition that consists of sub-propositions (e. g., “eat ice cream or don't drink beer”).

Examples for Building Blocks



If I don't **drink beer** to a meal, then I always **eat fish**. Whenever I **have fish** and **beer** with the same meal, I abstain from **ice cream**. When I **eat ice cream** or don't **drink beer**, then I never touch **fish**.

- Every sentence is a proposition that consists of sub-propositions (e. g., “eat ice cream or don't drink beer”).
- atomic propositions “**drink beer**”, “**eat fish**”, “**eat ice cream**”

Examples for Building Blocks



If I **don't** drink beer to a meal, **then** I always eat fish. **Whenever** I have fish **and** beer with the same meal, I **abstain** from ice cream. **When** I eat ice cream **or don't** drink beer, **then** I **never** touch fish.

- Every sentence is a proposition that consists of sub-propositions (e. g., “eat ice cream or don't drink beer”).
- atomic propositions “drink beer”, “eat fish”, “eat ice cream”
- logical connectives “**and**”, “**or**”, **negation**, “**if**, **then**”

Problems with Natural Language



If I don't drink beer to a meal, then I always eat fish.

Whenever I have fish and beer with the same meal, I abstain from ice cream.

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Problems with Natural Language



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Whenever I have fish and beer **with the same meal**, I abstain from ice cream.

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- **"irrelevant" information**

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- **different formulations for the same connective/proposition**

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- “irrelevant” information
- **different formulations for the same** connective/**proposition**

Problems with Natural Language



If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

- “irrelevant” information
- different formulations for the same connective/proposition

What is Next?

- What are meaningful (well-defined) sequences of atomic propositions and connectives?
“if then EatIceCream not or DrinkBeer and” not meaningful
→ **syntax**
- What does it mean if we say that a statement is true?
Is “DrinkBeer and EatFish” true?
→ **semantics**
- When does a statement logically follow from another?
Does “EatFish” follow from “if DrinkBeer, then EatFish”?
→ **logical entailment**
- **German:** Syntax, Semantik, logische Folgerung

Questions



Questions?

Syntax

Syntax of Propositional Logic

Definition (Syntax of Propositional Logic)

Let A be a set of **atomic propositions**. The set of **propositional formulas** (over A) is inductively defined as follows:

- Every **atom** $a \in A$ is a propositional formula over A .
- If φ is a propositional formula over A , then so is its **negation** $\neg\varphi$.
- If φ and ψ are propositional formulas over A , then so is the **conjunction** $(\varphi \wedge \psi)$.
- If φ and ψ are propositional formulas over A , then so is the **disjunction** $(\varphi \vee \psi)$.

The **implication** $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.

The **biconditional** $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.

German: atomare Aussage, aussagenlogische Formel, Atom, Negation, Konjunktion, Disjunktion, Implikation, Bikonditional

Syntax: Examples

Which of the following sequences of symbols are propositional formulas over the set of all possible letter sequences?

- $(A \wedge (B \vee C))$

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- $(A \wedge \neg(B \leftrightarrow)C)$

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- $(A \wedge \neg(B \leftrightarrow)C)$
- $(A \vee \neg(B \leftrightarrow C))$
- $((A \leq B) \wedge C)$

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- $((A \leq B) \wedge C)$
- $((A_1 \wedge A_2) \vee \neg(A_3 \leftrightarrow A_2))$

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Which kinds of formula are they (atom, conjunction, ...)?

Questions



Questions?

Semantics

Meaning of Propositional Formulas?

So far propositional formulas are only symbol sequences without any meaning.

For example, what does this mean:

$((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream})?$

▷ **We need semantics!**

Semantics of Propositional Logic

Definition (Semantics of Propositional Logic)

A **truth assignment** (or **interpretation**) for a set of atomic propositions A is a function $\mathcal{I} : A \rightarrow \{0, 1\}$.

A propositional **formula** φ (over A) **holds under** \mathcal{I} (written as $\mathcal{I} \models \varphi$) according to the following definition:

$\mathcal{I} \models a$	iff	$\mathcal{I}(a) = 1$	(for $a \in A$)
$\mathcal{I} \models \neg\varphi$	iff	not $\mathcal{I} \models \varphi$	
$\mathcal{I} \models (\varphi \wedge \psi)$	iff	$\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$	
$\mathcal{I} \models (\varphi \vee \psi)$	iff	$\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$	

Question: should we define semantics of $(\varphi \rightarrow \psi)$ and $(\varphi \leftrightarrow \psi)$?

German: Wahrheitsbelegung/Interpretation, φ gilt unter \mathcal{I}

Semantics of Propositional Logic: Terminology

- For $\mathcal{I} \models \varphi$ we also say \mathcal{I} is a model of φ and that φ is true under \mathcal{I} .
- If φ does not hold under \mathcal{I} , we write this as $\mathcal{I} \not\models \varphi$ and say that \mathcal{I} is no model of φ and that φ is false under \mathcal{I} .
- **Note:** \models is not part of the formula but part of the meta language (speaking about a formula).

German: \mathcal{I} ist ein/kein Modell von φ ; φ ist wahr/falsch unter \mathcal{I} ;

Metasprache

Semantics: Example (1)

$$A = \{\text{DrinkBeer}, \text{EatFish}, \text{EatIceCream}\}$$

$$\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$$

$$\varphi = (\neg \text{DrinkBeer} \rightarrow \text{EatFish})$$

Do we have $\mathcal{I} \models \varphi$?

Semantics: Example (2)

Goal: prove $\mathcal{I} \models \varphi$.

Let us use the definitions we have seen:

$$\begin{aligned} \mathcal{I} \models \varphi &\text{ iff } \mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish}) \\ &\text{ iff } \mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \text{EatFish}) \\ &\text{ iff } \mathcal{I} \models \neg\neg\text{DrinkBeer} \text{ or } \mathcal{I} \models \text{EatFish} \end{aligned}$$

This means that if we want to prove $\mathcal{I} \models \varphi$, it is sufficient to prove

$$\mathcal{I} \models \neg\neg\text{DrinkBeer}$$

or to prove

$$\mathcal{I} \models \text{EatFish}.$$

We attempt to prove the first of these statements.

Semantics: Example (3)

New goal: prove $\mathcal{I} \models \neg\neg\text{DrinkBeer}$.

We again use the definitions:

$$\begin{aligned} \mathcal{I} \models \neg\neg\text{DrinkBeer} &\text{ iff not } \mathcal{I} \models \neg\text{DrinkBeer} \\ &\text{ iff not not } \mathcal{I} \models \text{DrinkBeer} \\ &\text{ iff } \mathcal{I} \models \text{DrinkBeer} \\ &\text{ iff } \mathcal{I}(\text{DrinkBeer}) = 1 \end{aligned}$$

The last statement is true for our interpretation \mathcal{I} .

To write this up as a **proof** of $\mathcal{I} \models \varphi$,
we can go through this line of reasoning back-to-front,
starting from our assumptions and ending with the conclusion
we want to show.

Semantics: Example (4)

Let $\mathcal{I} = \{\text{DrinkBeer} \mapsto 1, \text{EatFish} \mapsto 0, \text{EatIceCream} \mapsto 1\}$.

Proof that $\mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$:

- (1) We have $\mathcal{I} \models \text{DrinkBeer}$
(uses defn. of \models for atomic props. and fact $\mathcal{I}(\text{DrinkBeer}) = 1$).
- (2) From (1), we get $\mathcal{I} \not\models \neg\text{DrinkBeer}$
(uses defn. of \models for negations).
- (3) From (2), we get $\mathcal{I} \models \neg\neg\text{DrinkBeer}$
(uses defn. of \models for negations).
- (4) From (3), we get $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \psi)$ for all formulas ψ ,
in particular $\mathcal{I} \models (\neg\neg\text{DrinkBeer} \vee \text{EatFish})$
(uses defn. of \models for disjunctions).
- (5) From (4), we get $\mathcal{I} \models (\neg\text{DrinkBeer} \rightarrow \text{EatFish})$
(uses defn. of “ \rightarrow ”).



Questions



Questions?

Properties of Propositional Formulas

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A propositional formula φ is

- **satisfiable** if φ has at least one model
- **unsatisfiable** if φ is not satisfiable
- **valid** (or a **tautology**) if φ is true under every interpretation
- **falsifiable** if φ is no tautology

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How can we show that a formula has one of these properties?

Examples

- Show that $(A \wedge B)$ is **satisfiable**.

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So far all proofs by specifying **one** interpretation.

How to prove that a given formula is valid/unsatisfiable/
not satisfiable/not falsifiable?

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So far all proofs by specifying **one** interpretation.

How to prove that a given formula is valid/unsatisfiable/
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\rightsquigarrow must consider **all possible** interpretations

Truth Tables

Evaluate for all possible interpretations
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$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models (A \wedge B)$
0	0	
0	1	
1	0	
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1	0	No
1	1	Yes

$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models (A \vee B)$
0	0	No
0	1	Yes
1	0	Yes
1	1	Yes

Truth Tables in General

Similarly in the case where we consider a formula whose building blocks are themselves arbitrary unspecified formulas:

$\mathcal{I} \models \varphi$	$\mathcal{I} \models \psi$	$\mathcal{I} \models (\varphi \wedge \psi)$
No	No	No
No	Yes	No
Yes	No	No
Yes	Yes	Yes

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No	No	No
No	Yes	No
Yes	No	No
Yes	Yes	Yes

Exercises: truth table for $(\varphi \rightarrow \psi)$

Truth Tables for Properties of Formulas

Is $\varphi = ((A \rightarrow B) \vee (\neg B \rightarrow A))$ valid, unsatisfiable, ...?

$\mathcal{I}(A)$	$\mathcal{I}(B)$	$\mathcal{I} \models \neg B$	$\mathcal{I} \models (A \rightarrow B)$	$\mathcal{I} \models (\neg B \rightarrow A)$	$\mathcal{I} \models \varphi$
0	0	Yes	Yes	No	Yes
0	1	No	Yes	Yes	Yes
1	0	Yes	No	Yes	Yes
1	1	No	Yes	Yes	Yes

Connection Between Formula Properties and Truth Tables

A propositional formula φ is

- **satisfiable** if φ has at least one model
 \rightsquigarrow result in at least one row is “Yes”
- **unsatisfiable** if φ is not satisfiable
 \rightsquigarrow result in all rows is “No”
- **valid** (or a **tautology**) if φ is true under every interpretation
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- **falsifiable** if φ is no tautology
 \rightsquigarrow result in at least one row is “No”

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2		4 interpretations (rows)
3		8 interpretations (rows)
n		??? interpretations

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n		2^n interpretations

Some examples: $2^{10} = 1024$, $2^{20} = 1048576$, $2^{30} = 1073741824$

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n		2^n interpretations

Some examples: $2^{10} = 1024$, $2^{20} = 1048576$, $2^{30} = 1073741824$

↪ not viable for larger formulas; we need a different solution

- more on difficulty of satisfiability etc.: Part E of this course
- practical algorithms: Foundations of AI course

Questions



Questions?

Summary

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- **propositional logic** based on atomic propositions
- **syntax** defines what well-formed formulas are
- **semantics** defines when a formula is true
- **interpretations** are the basis of semantics
- **satisfiability** and **validity** are important properties of formulas
- **truth tables** systematically consider all possible interpretations
- truth tables are only useful for small formulas