

Theory of Computer Science

A3. Proof Techniques

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February 22, 2017

Introduction

Mathematical Statements

Mathematical Statement

A **mathematical statement** consists of a set of **preconditions** and a set of **conclusions**.

The statement is **true** if the conclusions are true whenever the preconditions are true.

German: mathematische Aussage, Voraussetzung, Folgerung/Konklusion, wahr

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Notes:

- set of preconditions is sometimes empty
- often, “assumptions” is used instead of “preconditions”; slightly unfortunate because “assumption” is also used with another meaning (\rightsquigarrow cf. indirect proofs)

Examples of Mathematical Statements

Examples (some true, some false):

- “Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd.”
- “There exists an even prime number.”
- “Let $p \in \mathbb{N}_0$ with $p \geq 3$ be a prime number. Then p is odd.”
- “All prime numbers $p \geq 3$ are odd.”
- “For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ”
- “The equation $a^k + b^k = c^k$ has infinitely many solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 2$.”
- “The equation $a^k + b^k = c^k$ has no solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 3$.”

Which ones are true, which ones are false?

Proofs

Proof

A **proof** derives the correctness of a mathematical statement from a set of **axioms** and **previously proven statements**.

It consists of a sequence of **proof steps**, each of which **directly follows** from the axioms, previously proven statements and the preconditions of the statement, ending with the conclusions of the theorem.

German: Beweis, Axiom, Beweisschritt

Disproofs

- A **disproof** (**refutation**) analogously shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

German: Widerlegung

- Formally, disproofs are proofs of modified (“negated”) statements.
- Be careful about how to negate a statement!

Proof Strategies

typical proof/disproof strategies:

- 1 “All $x \in S$ with the property P also have the property Q .”
“For all $x \in S$: if x has property P , then x has property Q .”
 - To prove, assume you are given an arbitrary $x \in S$ that has the property P .
Give a sequence of proof steps showing that x must have the property Q .
 - To disprove, find a **counterexample**, i. e., find an $x \in S$ that has property P but not Q and prove this.

Proof Strategies

typical proof/disproof strategies:

- ② “ A is a subset of B .”
 - To prove, assume you have an arbitrary element $x \in A$ and prove that $x \in B$.
 - To disprove, find an element in $x \in A \setminus B$ and prove that $x \in A \setminus B$.

Proof Strategies

typical proof/disproof strategies:

- ③ “For all $x \in S$: x has property P iff x has property Q .”
(“iff”: “if and only if”)
 - To prove, separately prove “if P then Q ” and “if Q then P ”.
 - To disprove, disprove “if P then Q ” or disprove “if Q then P ”.
- German: “iff” = gdw. (“genau dann, wenn”)

Proof Strategies

typical proof/disproof strategies:

- ④ “ $A = B$ ”, where A and B are sets.
 - To prove, separately prove “ $A \subseteq B$ ” and “ $B \subseteq A$ ”.
 - To disprove, disprove “ $A \subseteq B$ ” or disprove “ $B \subseteq A$ ”.

Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contraposition
- mathematical induction
- structural induction

German: direkter Beweis, indirekter Beweis (Beweis durch Widerspruch), Kontraposition, vollständige Induktion, strukturelle Induktion

Direct Proof

Direct Proof

Direct Proof

Direct derivation of the statement by deducing or rewriting.

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

We first show that $x \in A \cap (B \cup C)$ implies
 $x \in (A \cap B) \cup (A \cap C)$ (“only-if” part, “ \Rightarrow ” part, “ \subseteq ” part):

...

German: Hin-Richtung

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Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

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German: Hin-Richtung

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Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

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Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

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Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.

If $x \in B$ then, because $x \in A$ is true, $x \in A \cap B$ must be true.

Otherwise, because $x \in B \cup C$ we know that $x \in C$ and thus with $x \in A$, that $x \in A \cap C$.

In both cases $x \in A \cap B$ or $x \in A \cap C$,
and we conclude $x \in (A \cap B) \cup (A \cap C)$

German: Hin-Richtung

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

“if” part, “ \Leftarrow ” part, \supseteq part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

...

German: Rückrichtung

Direct Proof: Example

Theorem (distributivity)

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Proof (continued).

“if” part, “ \Leftarrow ” part, \supseteq part: we must show that $x \in (A \cap B) \cup (A \cap C)$ implies $x \in A \cap (B \cup C)$.

Let $x \in (A \cap B) \cup (A \cap C)$.

If $x \in A \cap B$ then $x \in A$ and $x \in B$.

The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

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Let $x \in (A \cap B) \cup (A \cap C)$.

If $x \in A \cap B$ then $x \in A$ and $x \in B$.

The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.

This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

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Let $x \in (A \cap B) \cup (A \cap C)$.

If $x \in A \cap B$ then $x \in A$ and $x \in B$.

The latter implies $x \in B \cup C$ and hence $x \in A \cap (B \cup C)$.

If $x \notin A \cap B$ we know $x \in A \cap C$ due to $x \in (A \cap B) \cup (A \cap C)$.

This (analogously) implies $x \in A$ and $x \in C$, and hence $x \in B \cup C$ and thus $x \in A \cap (B \cup C)$.

In both cases we conclude $x \in A \cap (B \cup C)$.

...

German: Rückrichtung

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof (continued).

We have shown that every element of $A \cap (B \cup C)$ is an element of $(A \cap B) \cup (A \cap C)$ and vice versa. Thus, both sets are equal. □

Direct Proof: Example

Theorem (distributivity)

For all sets A, B, C : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.

Alternative:

$$\begin{aligned} A \cap (B \cup C) &= \{x \mid x \in A \text{ and } x \in B \cup C\} \\ &= \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\} \\ &= \{x \mid (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\} \\ &= \{x \mid x \in A \cap B \text{ or } x \in A \cap C\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$



Questions



Questions?

Indirect Proof

Indirect Proof

Indirect Proof (Proof by Contradiction)

- Make an **assumption** that the statement is false.
- Derive a **contradiction** from the assumption together with the preconditions of the statement.
- This shows that the assumption must be false given the preconditions of the statement, and hence the original statement must be true.

German: Annahme, Widerspruch

Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

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There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.



Indirect Proof: Example

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There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \dots \cdot p_n + 1$.



Indirect Proof: Example

Theorem

There are infinitely many prime numbers.

Proof.

Assumption: There are only finitely many prime numbers.

Let $P = \{p_1, \dots, p_n\}$ be the set of all prime numbers.

Define $m = p_1 \cdot \dots \cdot p_n + 1$.

Since $m \geq 2$, it must have a prime factor.

Let p be such a prime factor.



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Let p be such a prime factor.

Since p is a prime number, p has to be in P .



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Define $m = p_1 \cdot \dots \cdot p_n + 1$.

Since $m \geq 2$, it must have a prime factor.

Let p be such a prime factor.

Since p is a prime number, p has to be in P .

The number m is not divisible without remainder by any of the numbers in P . Hence p is no factor of m .

\rightsquigarrow **Contradiction**



Questions



Questions?

Contraposition

Contraposition

(Proof by) Contraposition

Prove “If A , then B ” by proving “If not B , then not A .”

German: (Beweis durch) Kontraposition

Contraposition

(Proof by) Contraposition

Prove “If A , then B ” by proving “If not B , then not A .”

German: (Beweis durch) Kontraposition

Examples:

- Prove “For all $n \in \mathbb{N}_0$: if n^2 is odd, then n is odd” by proving “For all $n \in \mathbb{N}_0$, if n is even, then n^2 is even.”
- Prove “For all $n \in \mathbb{N}_0$: if n is not a square number, then \sqrt{n} is irrational” by proving “For all $n \in \mathbb{N}_0$: if \sqrt{n} is rational, then n is a square number.”

Mathematical Induction

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Mathematical Induction

Proof of a statement for all natural numbers n with $n \geq m$

- **basis**: proof of the statement for $n = m$
- **induction hypothesis (IH)**:
suppose that statement is true for all k with $m \leq k \leq n$
- **inductive step**: proof of the statement for $n + 1$
using the induction hypothesis

German: vollständige Induktion, Induktionsanfang,
Induktionsvoraussetzung, Induktionsschritt

Mathematical Induction: Example I

Theorem

For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^n (2k - 1) = n^2$

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Proof.

Mathematical induction over n :

basis $n = 1$: $\sum_{k=1}^1 (2k - 1) = 2 - 1 = 1 = 1^2$



Mathematical Induction: Example I

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Proof.

Mathematical induction over n :

basis $n = 1$: $\sum_{k=1}^1 (2k - 1) = 2 - 1 = 1 = 1^2$

IH: $\sum_{k=1}^m (2k - 1) = m^2$ for all $1 \leq m \leq n$



Mathematical Induction: Example I

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For all $n \in \mathbb{N}_0$ with $n \geq 1$: $\sum_{k=1}^n (2k - 1) = n^2$

Proof.

Mathematical induction over n :

basis $n = 1$: $\sum_{k=1}^1 (2k - 1) = 2 - 1 = 1 = 1^2$

IH: $\sum_{k=1}^m (2k - 1) = m^2$ for all $1 \leq m \leq n$

inductive step $n \rightarrow n + 1$:

$$\begin{aligned}\sum_{k=1}^{n+1} (2k - 1) &= \left(\sum_{k=1}^n (2k - 1) \right) + 2(n + 1) - 1 \\ &\stackrel{\text{IH}}{=} n^2 + 2(n + 1) - 1 \\ &= n^2 + 2n + 1 = (n + 1)^2\end{aligned}$$



Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$ with prime numbers p_1, \dots, p_m .

Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$ with prime numbers p_1, \dots, p_m .

Proof.

Mathematical Induction over n :

basis $n = 2$: trivially satisfied, since 2 is prime

...

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Proof.

Mathematical Induction over n :

basis $n = 2$: trivially satisfied, since 2 is prime

IH: Every natural number k with $2 \leq k \leq n$
can be written as a product of prime numbers. ...

Mathematical Induction: Example II

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Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$ with prime numbers p_1, \dots, p_m .

Proof (continued).

inductive step $n \rightarrow n + 1$:

- Case 1: $n + 1$ is a prime number \rightsquigarrow trivial



Mathematical Induction: Example II

Theorem

Every natural number $n \geq 2$ can be written as a product of prime numbers, i. e. $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$ with prime numbers p_1, \dots, p_m .

Proof (continued).

inductive step $n \rightarrow n + 1$:

- Case 1: $n + 1$ is a prime number \rightsquigarrow trivial
- Case 2: $n + 1$ is not a prime number.

There are natural numbers $2 \leq q, r \leq n$ with $n + 1 = q \cdot r$.

Using IH shows that there are prime numbers

q_1, \dots, q_s with $q = q_1 \cdot \dots \cdot q_s$ and

r_1, \dots, r_t with $r = r_1 \cdot \dots \cdot r_t$.

Together this means $n + 1 = q_1 \cdot \dots \cdot q_s \cdot r_1 \cdot \dots \cdot r_t$.



Structural Induction

Inductively Defined Sets: Examples

Example (Natural Numbers)

The set \mathbb{N}_0 of natural numbers is inductively defined as follows:

- 0 is a natural number.
- If n is a natural number, then $n + 1$ is a natural number.

Inductively Defined Sets: Examples

Example (Natural Numbers)

The set \mathbb{N}_0 of natural numbers is inductively defined as follows:

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Example (Binary Tree)

The set \mathcal{B} of binary trees is inductively defined as follows:

- \square is a binary tree (a **leaf**)
- If L and R are binary trees, then $\langle L, \bigcirc, R \rangle$ is a binary tree (with **inner node** \bigcirc).

German: Binärbaum, Blatt, innerer Knoten

Inductively Defined Sets: Examples

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German: Binärbaum, Blatt, innerer Knoten

Implicit statement: all elements of the set can be constructed by finite application of these rules

Inductive Definition of a Set

Inductive Definition

A set M can be defined **inductively** by specifying

- **basic elements** that are contained in M
- **construction rules** of the form
“Given some elements of M , another element of M can be constructed like this.”

German: induktive Definition, Basiselemente, Konstruktionsregeln

Structural Induction

Structural Induction

Proof of statement for all elements of an inductively defined set

- **basis**: proof of the statement for the basic elements
- **induction hypothesis (IH)**:
suppose that statement is true for some elements M
- **inductive step**: proof of the statement for elements constructed by applying a construction rule to M
(one inductive step for each construction rule)

German: strukturelle Induktion, Induktionsanfang, Induktionsvoraussetzung, Induktionsschritt

Structural Induction: Example (1)

Definition (Leaves of a Binary Tree)

The number of **leaves** of a binary tree B , written $leaves(B)$, is defined as follows:

$$leaves(\square) = 1$$

$$leaves(\langle L, \circlearrowleft, R \rangle) = leaves(L) + leaves(R)$$

Definition (Inner Nodes of a Binary Tree)

The number of **inner nodes** of a binary tree B , written $inner(B)$, is defined as follows:

$$inner(\square) = 0$$

$$inner(\langle L, \circlearrowleft, R \rangle) = inner(L) + inner(R) + 1$$

Structural Induction: Example (2)

Theorem

For all binary trees B : $inner(B) = leaves(B) - 1$.

Structural Induction: Example (2)

Theorem

For all binary trees B : $inner(B) = leaves(B) - 1$.

Proof.

induction basis:

$$inner(\square) = 0 = 1 - 1 = leaves(\square) - 1$$

\rightsquigarrow statement is true for base case

...

Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \circlearrowleft, R \rangle$, we may use that it is true for the subtrees L and R .



Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree $\langle L, \circlearrowleft, R \rangle$, we may use that it is true for the subtrees L and R .

inductive step for $B = \langle L, \circlearrowleft, R \rangle$:

$$\begin{aligned} \text{inner}(B) &= \text{inner}(L) + \text{inner}(R) + 1 \\ &\stackrel{\text{IH}}{=} (\text{leaves}(L) - 1) + (\text{leaves}(R) - 1) + 1 \\ &= \text{leaves}(L) + \text{leaves}(R) - 1 = \text{leaves}(B) - 1 \end{aligned}$$



Questions



Questions?

Summary

Summary

- A **proof** is based on axioms and previously proven statements.
- Individual **proof steps** must be obvious derivations.
- **direct proof**: sequence of derivations or rewriting
- **indirect proof**: refute the negated statement
- **contraposition**: prove " $A \Rightarrow B$ " as " $\text{not } B \Rightarrow \text{not } A$ "
- **mathematical induction**: prove statement for a starting point and show that it always carries over to next number
- **structural induction**: generalization of mathematical induction to arbitrary recursive structures

Preparation for the Next Lecture

What's the secret of your long life?



I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Simplify this advice!